F-theory with Quivers

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Based on work in progress with A. Collinucci, M. Fazzi and D. Morrison

Introduction

- In the M/F-theory geometric engineering, Abelian gauge symmetries emanate from reduction of C_3 along harmonic, normalizable 2-forms.

$$C_3 \sim A_\mu dx^\mu \wedge \omega$$

- The 2-form ω can be described via its Poincaré dual cycle (divisor).
- In F-theory, the elliptically fibered CY has extra sections, which are identified as new divisor classes giving rise to U(1)s. [Morrison,Vafa]

  Techniques expanded and refined over the past few years.

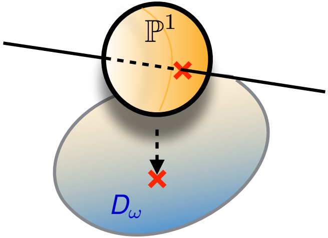
  [Grimm,Weigand; Morrison,Park; (Borchmann),Mayrhofer,Palti,Weigand; Cvetic,(Grassi),Klevers,Piragua,(Song),(Taylor); V.Braun,Grimm,Keitel; A.Braun,Collinucci,RV]
- In this talk, new way of detecting such divisors in varieties that admit small resolutions. We will focus on CY *three-fold*.

Introduction

Simplest way to get a massless U(1) in F-theory is 'U(1) restriction':
[Grimm,Weigand]

$$(y - a_3)(y + a_3) = s(s^2 + b_2 s + 2b_4) \qquad \text{where } s \equiv x - \frac{b_2}{3}$$

- elliptic fibration has one conifold singularity at (s, y, a_3, b_4);
- after small resolution one sees the extra divisor D_ω that intersects the exceptional $\mathbb{P}^1$ at one point.



M2 couples to the 3-form:

$$\int_{M2} C_3 = \int A_\mu dx^\mu \int_{\mathbb{P}^1} \omega = (\mathbb{P}^1 \cdot D_\omega) \int A_\mu dx^\mu$$

Introduction

'U(1) restriction' is so far also the only case where a **matrix factorization (MF)** has been worked out in F-theory. [Collinucci,Savelli]

This formalism allows to deal with **singular** manifolds without resolution.

- In particular, a 'line bundle' M on CY arises naturally. $c_1(M) \sim \omega$ related to the U(1) divisor.
- Identify massless matter charged under this U(1).

Moreover, MF comes naturally with (NC) resolution and associated **quiver**. [Aspinwall,Morrison]

Apply this formalism to more generic case with abelian gauge symmetries. This approach can give new insights.

Conifold

In U(1) restriction, Weierstrass model factorizes as

$$(y - a_3)(y + a_3) = s(s^2 + b_2 s + 2b_4) \qquad \text{where } s \equiv x - \tfrac{b_2}{3}$$

$$y_+ y_- = s\, w$$

with $y_\pm = y \pm a_3$, $w = s^2 + b_2 s + 2b_4$

- ⋇ Non-Cartier divisors $(y_\pm, s) \leftrightarrow$ extra section of elliptic fibration.
- ⋇ After small resolution: $(y_\pm, s)$ become Cartier divisors; one sees exceptional $\mathbb{P}^1$ wrapped by M2 (charged states)

Weak coupling limit: one U(1) brane and its orientifold image, intersecting away from O7 (where massless matter live).

Conifold - Matrix Factorization (MF)

Eq $y_+ y_- = s\,w$ admits a (pair of) **MF**, i.e. a pair of matrices (ϕ, ψ) s.t.

$$\phi \cdot \psi = \psi \cdot \phi = (y_+ y_- - s\,w)\mathbf{1}_2$$

For the conifold:

$$\phi = \begin{pmatrix} y_- & s \\ w & y_+ \end{pmatrix} \qquad \psi = \begin{pmatrix} y_+ & -s \\ -w & y_- \end{pmatrix}$$

From ϕ, ψ one can define (MCM) modules over R [Eisenbud], e.g.

$$M = \mathrm{coker}(R^{\oplus 2} \xrightarrow{\psi} R^{\oplus 2}) \approx R^{\oplus 2} / \mathrm{Im}\psi$$

(where $R = \mathbb{C}[y_+, y_-, s, w]/(y_+ y_- - s\,w)$ is the coordinate ring.)

$\rightsquigarrow$ 'Line bundle' over conifold (but 'rank two' on sing locus)

- defined on sing space
- $c_1 \sim$ extra div $\leftrightarrow$ massless U(1).

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$M = \text{coker}(R^{\otimes 2} \xrightarrow{\psi} R^{\otimes 2}) \sim$ Line bundle over conifold (except on sing locus)

- $c_1(M) \sim$ locus where a generic section vanishes.

- coker $\psi \cong$ Im $\phi \quad \rightarrow \quad c_1 : \begin{pmatrix} y_- & s \\ w & y_+ \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = 0,$

  i.e. $\quad \sigma_1 y_- + \sigma_2 s = 0\,, \qquad \sigma_1 w + \sigma_2 y_+ = 0$

- Family of non-Cartier divisors, among which extra-section of elliptic fibration ($\sigma_1 = 0$, $\sigma_2 = 1$)..

Conifold - Non Commutative Resolution (NCCR)

NCCR: enlarge coordinate ring $R = \mathbb{C}[y_+, y_-, s, w]/(y_+ y_- - s\,w)$
by replacing it with

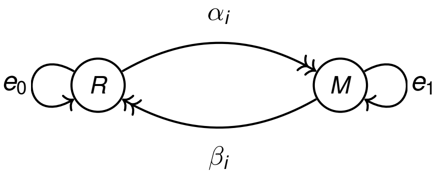
$$A \equiv \mathrm{End}_R(R \oplus M),$$

i.e.

$$\begin{matrix} \mathrm{Hom}(R,R) & \oplus & \mathrm{Hom}(M,M) & \oplus & \mathrm{Hom}(R,M) & \oplus & \mathrm{Hom}(M,R) \\ \cong R & & \cong R & & \cong M & & \cong \widetilde{M} \\ e_0 & & e_1 & & \alpha_i & & \beta_i \end{matrix}$$

This is a (in principle) non-coomutative ring.

Associated quiver:



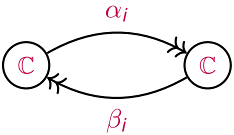
Conifold - Quiver rep and NCCR

Given CY X, equivalence of Categories

{ coherent sheaves } $\longleftrightarrow$ { A-modules } $\longleftrightarrow$ { representations of quiver }

Physically, objects are D-branes on X. (E.g. take IIA string on X.)

- Moduli space of D0-branes is the full space X. Its quiver rep $\vec{d} = (1, 1)$



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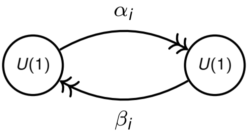

- Moduli space: all possible α_i, β_i modulo relative $U(1)$ and D-term

α_1	α_2	β_1	β_2
1	1	-1	-1

with $\quad |\alpha_1|^2 + |\alpha_2|^2 - |\beta_1|^2 - |\beta_2|^2 = \theta$

- Exceptional $\mathbb{P}^1$ is $\beta_1 = \beta_2 = 0$ (for $\theta > 0$).
- In resolved space, **extra divisor** is zero locus of a section of M, i.e.

$\sigma_1\alpha_1 + \sigma_2\alpha_2 = 0 \qquad$ intersects exceptional $\mathbb{P}^1$ in one point.

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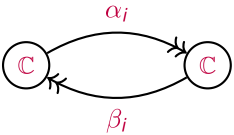

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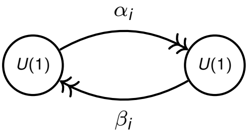

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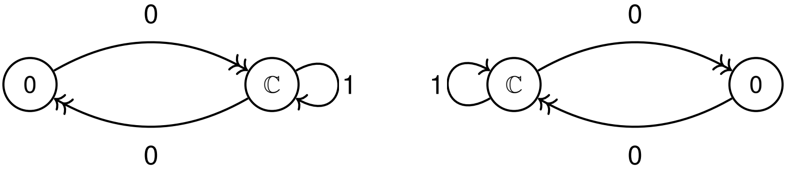
Conifold - Quiver rep

D0-brane splits into fractional branes (D2 wrapping vanishing $\mathbb{P}^1$): $\vec{d} = (1,1)$ rep splits into simple reps, i.e. $\vec{d} = (0,1)$ and $\vec{d} = (1,0)$:



- $*$ No moduli space: D2s wrap rigid curve.
- $*$ BPS particles in space-time.

  In the singular limit, they become massless for given choice of B-field (not both) and are charged under $A_\mu \sim \int_{\mathbb{P}^1} C_3$.
  $\rightarrow$ massless M2 charged states in M-theory language.

Morrison-Park

U(1) restriction is subcase of class of ellitpic fibrations with *one* extra section.

Full class is described by Morrison-Park

$$y^2 = s^3 + c_2 s^2 + \left(c_1 c_3 - b^2 c_0\right) s + c_0 c_3^2 - b^2 c_0 c_2 + \frac{b^2 c_1^2}{4}$$

- A rational section $\hookrightarrow$ massless U(1);
- two curves of conifold-like sing $\hookrightarrow$ charged matter

(At weak coupling: pair of brane-imagebrane and invariant brane.)

Is there a 2×2 MF ?

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Morrison-Park

The answer is NO !

But $\exists$ 4 $\times$ 4 MF:

$$\Psi_{MP} = \begin{pmatrix} y + \frac{c_1 b}{2} & s & -c_3 & -b \\ c_1 c_3 + s(s + c_2) & y - \frac{c_1 b}{2} & -b(s + c_2) & -c_3 \\ -c_0 c_3 & c_0 b & y + \frac{c_1 b}{2} & -s \\ c_0 b(s + c_2) & -c_0 c_3 & -c_1 c_3 - s(s + c_2) & y - \frac{c_1 b}{2} \end{pmatrix}$$

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How can we extract massless U(1) and charged matter?

Specialize to a simpler example:

$$c_0 \equiv -w, \qquad c_1 \equiv 0, \qquad c_2 \equiv 0, \qquad c_3 \equiv z, \qquad b \equiv w$$

get Laufer's threefold (also $s \to -s$ and $y \to -y$)

$$y^2 + s^3 + w^3 s + z^2 w = 0$$

singular at (y, s, z, w).

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singular at (y, s, z, w).

4×4 MF: [Curto,Morrison; Aspinwall,Morrison]

$$\Psi_L = \begin{pmatrix} y & s & z & w \\ -s^2 & y & -s\,w & z \\ -w\,z & w^2 & y & -s \\ -s\,w^2 & -w\,z & s^2 & y \end{pmatrix}$$

$$y^2 + s^3 + w^3 s + z^2 w = 0$$

- $M = \mathrm{coker}(R^{\otimes 4} \xrightarrow{\Psi_L} R^{\otimes 4})$ is now rank 2.

  Claim is that also in this case $c_1(M)$ gives extra divisor, i.e. new massless U(1) gauge boson.

- Matter at sing, i.e. when Ψ_L becomes zero rank.

Laufer

$$\Psi_L = \begin{pmatrix} y & s & z & w \\ -s^2 & y & -s\,w & z \\ -w\,z & w^2 & y & -s \\ -s\,w^2 & -w\,z & s^2 & y \end{pmatrix}$$

$M = \text{coker}(R^{\otimes 4} \xrightarrow{\Psi_L} R^{\otimes 4}) \sim$ Vector bndl over Laufer (except on sing locus)

- ⋇ $c_1(M) \sim$ locus where a two sections become parallel.

- ⋇ coker $\Psi_L \cong \text{Im}\,\Phi_L \quad \rightarrow \quad$ with specific choice of sections, c_1 :

$$z^2 + w^2 s = 0\,, \qquad s\,z + w\,y = 0\,, \qquad y\,z - w\,s^2 = 0\,,$$

  intersected with $y^2 + s^3 + w\,z^2 + s\,w^3 = 0$.

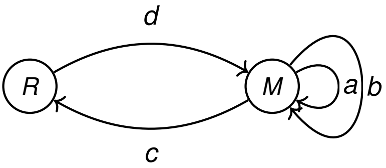
- ⋇ In general, family of non-Cartier divisors, among which extra (rational) section of elliptic fibration, i.e.

$$y = \frac{z^3}{w^3} \qquad\qquad s = -\frac{z^2}{w^2}$$

Again NCCR: enlarge coordinate ring $R = \mathbb{C}[y,s,z,w]/(y^2+s^3+w^3s+z^2w)$ by replacing it with

$$A \equiv \text{End}_R(R \oplus M),$$

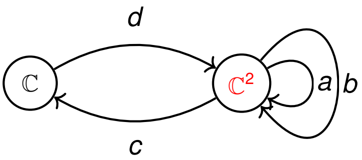
Associated quiver:



with relations

$$(b^2+dc)d = 0, \qquad c(b^2+dc) = 0, \qquad ab+ba = 0, \qquad a^2+bdc+dcb+b^3 = 0.$$

Full resolved space given by moduli space of D0 ($\vec{d} = (1,2)$ rep):



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Resolved space given by

- all possible values of maps a, b, c, d (except 'SR-ideal'),
- subject to realtions and
- modded by gauge transformations $U(1) \times U(2)$ and D-terms:

$$d^\dagger d - cc^\dagger = 2\theta \qquad dd^\dagger - c^\dagger c + [a, a^\dagger] + [b, b^\dagger] = \theta\, \mathbf{1}_2$$

Exceptional $\mathbb{P}^1$: locus that is pushed down to (y, s, w, z) in sing space.

For both phases:

$$a^2 = 0, \qquad b^2 = 0, \qquad ab + ba = 0$$

and

$$c = 0 \ \text{ for } \ \theta > 0\,, \qquad \text{while} \qquad d = 0 \ \text{ for } \ \theta < 0$$

Interpolate between the two phases by taking

$$a = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} \sqrt{2}\gamma & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ \sqrt{2}\delta \end{pmatrix}$$

D-terms become

$$|\delta|^2 - |\gamma|^2 = \theta\,, \qquad |\alpha|^2 + |\beta|^2 = |\delta|^2 + |\gamma|^2$$

Now, take e.g. phase $\theta > 0$. Then $\gamma = 0$ and $\mathbb{P}^1$ spanned by α, β with residual gauge transformation $(\alpha, \beta) \mapsto (\lambda\alpha, \lambda\beta)$.

Changing continuously θ, follow the flop transition through the singularity.

Laufer - U(1) divisor

Consider phase $\theta > 0$.

Extra divisor given by locus where two sections of M become parallel:

$$\sigma_1(ad \wedge d) + \sigma_2(bd \wedge d) + \sigma_3(abd \wedge d) = 0.$$

(Sections of M are d, ad, bd, abd.)

- Again family of divisors intersecting the $\mathbb{P}^1$ at one point: $\sigma_1\alpha + \sigma_2\beta = 0$.
- It corresponds to what one finds working in the sing space:

$$\begin{pmatrix} -sw^2 - z^2 & -yw + sz & -ws^2 - xz \\ yw + zs & -w^3 - s^2 & -zw^2 + ys \\ s^2w - yz & zw^2 + ys & -sw^3 - y^2 \end{pmatrix} \cdot \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = 0$$

(Before we specialized $\vec{\sigma} = (1, 0, 0)$.)

Laufer - Fractional branes

Fractional branes are $D0_R$ with $\vec{d} = (0,1)$ and $D0_L$ with $\vec{d} = (1,0)$.

Mobile D0 splits into $D0_L \oplus 2\, D0_R$.

- $D0_R$ has D2-charge equal to 1.
- $D0_R$ has D2-charge equal to 2.

Charge-1 fractional brane gives massless charged state in the singular limit. [Aspinwall,Morrison]

If it worked like in the conifold, changing the B field the charge-2 state may become massless. (Possibility to have double charge states.)

Conclusions

- Matrix factorization and quiver for fourfolds with massless U(1).
- Naturally encode extra U(1) divisor (already in the singular limit): family of representatives that includes extra section of elliptic fibration.
- Associated quiver gives resolution. Exceptional $\mathbb{P}^1$. Flop transition.
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- Higher charge states?
- Results presented here for three-fold. Four-fold at the edge of math research.

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The End