F-theory with Quivers

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Based on work in progress with A. Collinucci, M. Fazzi and D. Morrison
In the M/F-theory geometric engineering, Abelian gauge symmetries emanate from reduction of $C_3$ along harmonic, normalizable 2-forms.

\[ C_3 \sim A_\mu \, dx^\mu \wedge \omega \]

The 2-form $\omega$ can be described via its Poincaré dual cycle (divisor).

In F-theory, the elliptically fibered CY has extra sections, which are identified as new divisor classes giving rise to U(1)s. [Morrison, Vafa]

Techniques expanded and refined over the past few years.

[Morrison, Weigand; Morrison, Park; (Borchmann), Mayrhofer, Palti, Weigand; Cvetic, (Grassi), Klevers, Piragua, (Song), (Taylor); V. Braun, Grimm, Keitel; A. Braun, Collinucci, RV]

In this talk, new way of detecting such divisors in varieties that admit small resolutions. We will focus on CY *three-fold*. 

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F-theory with Quivers
Simplest way to get a massless U(1) in F-theory is ‘U(1) restriction’:
\[ (y - a_3)(y + a_3) = s(s^2 + b_2 s + 2b_4) \quad \text{where} \quad s \equiv x - \frac{b_2}{3} \]

- elliptic fibration has one conifold singularity at \((s, y, a_3, b_4)\);
- after small resolution one sees the extra divisor \(D_\omega\) that intersects the exceptional \(\mathbb{P}^1\) at one point.

\[
\int_{M^2} C_3 = \int A_\mu dx^\mu \int_{\mathbb{P}^1} \omega = (\mathbb{P}^1 \cdot D_\omega) \int A_\mu dx^\mu
\]
‘U(1) restriction’ is so far also the only case where a **matrix factorization (MF)** has been worked out in F-theory. [Collinucci, Savelli]

This formalism allows to deal with **singular** manifolds without resolution.

- In particular, a ‘line bundle’ \( M \) on CY arises naturally.
  \[ c_1(M) \sim \omega \] related to the U(1) divisor.
- Identify massless matter charged under this U(1).

Moreover, MF comes naturally with (NC) resolution and associated **quiver**. [Aspinwall, Morrison]

Apply this formalism to more generic case with abelian gauge symmetries. This approach can give new insights.
In U(1) restriction, Weierstrass model factorizes as:

$$(y - a_3)(y + a_3) = s(s^2 + b_2s + 2b_4)$$

where $s \equiv x - \frac{b_2}{3}$

with $y_\pm = y \pm a_3$, $w = s^2 + b_2s + 2b_4$

※ Non-Cartier divisors $(y_\pm, s) \leftrightarrow$ extra section of elliptic fibration.

※ After small resolution: $(y_\pm, s)$ become Cartier divisors; one sees exceptional $\mathbb{P}^1$ wrapped by M2 (charged states)

Weak coupling limit: one U(1) brane and its orientifold image, intersecting away from O7 (where massless matter live).
Eq $y_+y_- = s\, w$ admits a (pair of) MF, i.e. a pair of matrices $(\phi, \psi)$ s.t.

$$\phi \cdot \psi = \psi \cdot \phi = (y_+y_- - s\, w)1_2$$

For the conifold:

$$\phi = \begin{pmatrix} y_- & s \\ w & y_+ \end{pmatrix} \quad \psi = \begin{pmatrix} y_+ & -s \\ -w & y_- \end{pmatrix}$$

From $\phi, \psi$ one can define (MCM) modules over $R$ [Eisenbud], e.g.

$$M = \text{coker}(R^2 \xrightarrow{\psi} R^2) \cong R^2 / \text{Im}\psi$$

(where $R = \mathbb{C}[y_+, y_-, s, w]/(y_+y_- - s\, w)$ is the coordinate ring.)

$\leadsto$ ‘Line bundle’ over conifold (but ‘rank two’ on sing locus)

- defined on sing space
- $c_1 \sim$ extra div $\leftrightarrow$ massless U(1).
Eq \( y_+ y_- = s w \) admits a (pair of) **MF**, i.e. a pair of matrices \((\phi, \psi)\) s.t.

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\phi \cdot \psi = \psi \cdot \phi = (y_+ y_- - s w)1_2
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\]

From \(\phi, \psi\) one can define (MCM) modules over \(R\) [Eisenbud], e.g.

\[
M = \text{coker}(R^{\oplus 2} \xrightarrow{\psi} R^{\oplus 2}) \approx R^{\oplus 2} / \text{Im} \psi
\]

(\(R = \mathbb{C}[y_+, y_-, s, w]/(y_+y_- - s w)\) is the coordinate ring.)

\(\rightsquigarrow\) ‘Line bundle’ over conifold (but ‘rank two’ on sing locus)

- defined on sing space
- \(c_1 \sim\) extra div \(\leftrightarrow\) massless U(1).
For the conifold:

\[ \phi = \begin{pmatrix} y_- & s \\ w & y_+ \end{pmatrix} \quad \psi = \begin{pmatrix} y_+ & -s \\ -w & y_- \end{pmatrix} \]

\[ M = \text{coker}(R^2 \xrightarrow{\psi} R^2) \sim \text{Line bundle over conifold (except on sing locus)} \]

- \( c_1(M) \sim \text{locus where a generic section vanishes.} \)

- \( \text{coker } \psi \cong \text{Im } \phi \rightarrow c_1 : \begin{pmatrix} y_- & s \\ w & y_+ \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = 0, \)

  i.e. \( \sigma_1 y_- + \sigma_2 s = 0, \quad \sigma_1 w + \sigma_2 y_+ = 0 \)

- \( \text{Family of non-Cartier divisors, among which extra-section of elliptic fibration } (\sigma_1 = 0, \sigma_2 = 1). \)
**Conifold - Non Commutative Resolution (NCCR)**

**NCCR**: enlarge coordinate ring $R = \mathbb{C}[y_+, y_-, s, w]/(y_+ y_- - s w)$ by replacing it with

$$A \equiv \text{End}_R(R \oplus M),$$

i.e.

$$\begin{align*}
\text{Hom}(R, R) &\cong R \\
e_0 &\quad \cong R \\
\text{Hom}(M, M) &\cong R \\
e_1 &\quad \cong M \\
\text{Hom}(R, M) &\cong M \\
\alpha_i &\quad \cong \tilde{M} \\
\text{Hom}(M, R) &\cong \tilde{M} \\
\beta_i &
\end{align*}$$

This is a (in principle) non-coomutative ring.

**Associated quiver:**

![Quiver Diagram]

Roberto Valandro | F-theory with Quivers
Given CY $X$, equivalence of Categories

$\{ \text{coherent sheaves} \} \leftrightarrow \{ \text{A-modules} \} \leftrightarrow \{ \text{representations of quiver} \}$

Physically, objects are D-branes on $X$. (E.g. take IIA string on $X$.)

- Moduli space of D0-branes is the full space $X$. Its quiver rep $\vec{d} = (1, 1)$

- Moduli space: all possible $\alpha_i, \beta_i$ modulo relative $U(1)$ and D-term

$$\begin{align*}
\alpha_1 & \quad \alpha_2 & \quad \beta_1 & \quad \beta_2 \\
1 & \quad 1 & \quad -1 & \quad -1
\end{align*}$$

with $|\alpha_1|^2 + |\alpha_2|^2 - |\beta_1|^2 - |\beta_2|^2 = \theta$

- Exceptional $\mathbb{P}^1$ is $\beta_1 = \beta_2 = 0$ (for $\theta > 0$).

- In resolved space, **extra divisor** is zero locus of a section of $M$, i.e.

$$\sigma_1 \alpha_1 + \sigma_2 \alpha_2 = 0$$

intersects exceptional $\mathbb{P}^1$ in one point.
Given CY $X$, equivalence of Categories

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\{ \text{coherent sheaves} \} \longleftrightarrow \{ \text{A-modules} \} \longleftrightarrow \{ \text{representations of quiver} \}
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Physically, objects are D-branes on $X$. (E.g. take IIA string on $X$.)

- Moduli space of D0-branes is the full space $X$. Its quiver rep $\tilde{\mathbf{d}} = (1, 1)$

\[
\begin{array}{c}
\alpha_i \\
\mathbb{C} \\
\beta_i \\
\mathbb{C}
\end{array}
\quad \text{or in physics language} \quad
\begin{array}{c}
\alpha_i \\
U(1) \\
\beta_i \\
U(1)
\end{array}
\]

- Moduli space: all possible $\alpha_i, \beta_i$ modulo relative $U(1)$ and D-term

\[
\begin{array}{c}
\alpha_1 \\
1 \\
\beta_1 \\
-1
\end{array} \quad \begin{array}{c}
\alpha_2 \\
1 \\
\beta_2 \\
-1
\end{array} \quad \text{with} \quad |\alpha_1|^2 + |\alpha_2|^2 - |\beta_1|^2 - |\beta_2|^2 = \theta
\]

- Exceptional $\mathbb{P}^1$ is $\beta_1 = \beta_2 = 0$ (for $\theta > 0$).
- In resolved space, **extra divisor** is zero locus of a section of $M$, i.e.

\[
\sigma_1 \alpha_1 + \sigma_2 \alpha_2 = 0 \quad \text{intersects exceptional } \mathbb{P}^1 \text{ in one point.}
\]
D0-brane splits into fractional branes (D2 wrapping vanishing $\mathbb{P}^1$): $\vec{d} = (1, 1)$
rep splits into simple reps, i.e. $\vec{d} = (0, 1)$ and $\vec{d} = (1, 0)$:

* No moduli space: D2s wrap rigid curve.
* BPS particles in space-time.
  In the singular limit, they become massless for given choice of B-field (not both) and are charged under $A_\mu \sim \int_{\mathbb{P}^1} C_3$.
  $\rightarrow$ massless M2 charged states in M-theory language.
U(1) restriction is subcase of class of elliptic fibrations with one extra section.

Full class is described by Morrison-Park

\[ y^2 = s^3 + c_2 s^2 + \left( c_1 c_3 - b^2 c_0 \right) s + c_0 c_3^2 - b^2 c_0 c_2 + \frac{b^2 c_1^2}{4} \]

- A rational section ↦ massless U(1);
- two curves of conifold-like sing ↦ charged matter

(At weak coupling: pair of brane-imagebrane and invariant brane.)

Is there a 2 × 2 MF?
U(1) restriction is subcase of class of elliptic fibrations with one extra section.

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* A rational section \( \hookrightarrow \) massless U(1);
* two curves of conifold-like sing \( \hookrightarrow \) charged matter

(At weak coupling: pair of brane-imagebrane and invariant brane.)

Is there a \( 2 \times 2 \) MF?
The answer is **NO!**

But \( \exists 4 \times 4 \) MF:

\[
\Psi_{\text{MP}} = \begin{pmatrix}
  y + \frac{c_1 b}{2} & s & -c_3 & -b \\
  c_1 c_3 + s(s + c_2) & y - \frac{c_1 b}{2} & -b(s + c_2) & -c_3 \\
  -c_0 c_3 & c_0 b & y + \frac{c_1 b}{2} & -s \\
  c_0 b(s + c_2) & -c_0 c_3 & -c_1 c_3 - s(s + c_2) & y - \frac{c_1 b}{2}
\end{pmatrix}
\]

\[
y^2 = s^3 + c_2 s^2 + \left(c_1 c_3 - b^2 c_0\right) s + c_0 c_3^2 - b^2 c_0 c_2 + \frac{b^2 c_1^2}{4}
\]

How can we extract massless U(1) and charged matter?

Specialize to a simpler example:

\[
c_0 \equiv -w, \quad c_1 \equiv 0, \quad c_2 \equiv 0, \quad c_3 \equiv z, \quad b \equiv w
\]

get Laufer’s threefold (also \( s \rightarrow -s \) and \( y \rightarrow -y \))

\[
y^2 + s^3 + w^3 s + z^2 w = 0
\]

singular at \((y, s, z, w)\).
The answer is NO!

But $\exists$ $4 \times 4$ MF:

$$
\psi_{MP} = \begin{pmatrix}
  y + \frac{c_1 b}{2} & s & -c_3 & -b \\
  c_1 c_3 + s(s + c_2) & y - \frac{c_1 b}{2} & -b(s + c_2) & -c_3 \\
  -c_0 c_3 & c_0 b & y + \frac{c_1 b}{2} & -s \\
  c_0 b(s + c_2) & -c_0 c_3 & -c_1 c_3 - s(s + c_2) & y - \frac{c_1 b}{2}
\end{pmatrix}
$$

$$
y^2 = s^3 + c_2 s^2 + \left( c_1 c_3 - b^2 c_0 \right) s + c_0 c_3^2 - b^2 c_0 c_2 + \frac{b^2 c_1^2}{4}
$$

How can we extract massless U(1) and charged matter?

Specialize to a simpler example:

$c_0 \equiv -w$, $c_1 \equiv 0$, $c_2 \equiv 0$, $c_3 \equiv z$, $b \equiv w$

get Laufer's threefold (also $s \rightarrow -s$ and $y \rightarrow -y$)

$$
y^2 + s^3 + w^3 s + z^2 w = 0
$$

singular at $(y, s, z, w)$. 
$\Psi_L = \begin{pmatrix} \ y & s & z & w \\ -s^2 & y & -sw & z \\ -wz & w^2 & y & -s \\ -sw^2 & -wz & s^2 & y \end{pmatrix}$

$y^2 + s^3 + w^3 s + z^2 w = 0$

$M = \text{coker}(R^4 \xrightarrow{\Psi_L} R^4)$ is now rank 2.

Claim is that also in this case $c_1(M)$ gives extra divisor, i.e. new massless U(1) gauge boson.

Matter at sing, i.e. when $\Psi_L$ becomes zero rank.
\[ \Psi_L = \begin{pmatrix}
  y & s & z & w \\
  -s^2 & y & -sw & z \\
  -wz & w^2 & y & -s \\
  -sw^2 & -wz & s^2 & y
\end{pmatrix} \]

\[ M = \text{coker}(R^4 \xrightarrow{\Psi_L} R^4) \sim \text{Vector bndl over Laufer (except on sing locus)} \]

\begin{itemize}
  \item \( c_1(M) \sim \text{locus where two sections become parallel.} \)
  \item \( \text{coker } \Psi_L \cong \text{Im } \Phi_L \rightarrow \text{with specific choice of sections, } c_1 : \)
    \[ z^2 + w^2 s = 0, \quad s z + w y = 0, \quad y z - w s^2 = 0, \]
    intersected with \( y^2 + s^3 + w z^2 + s w^3 = 0. \)
  \item In general, \text{family of non-Cartier divisors}, among which \text{extra (rational)}
    section of elliptic fibration, i.e.
    \[ y = \frac{z^3}{w^3}, \quad s = -\frac{z^2}{w^2} \]
\end{itemize}
Again **NCCR**: enlarge coordinate ring \( R = \mathbb{C}[y, s, z, w]/(y^2 + s^3 + w^3 s + z^2 w) \) by replacing it with

\[
A \equiv \text{End}_R(R \oplus M),
\]

Associated quiver:

\[
\begin{align*}
\text{with relations} & \quad (b^2 + dc)d = 0, \quad c(b^2 + dc) = 0, \quad ab + ba = 0, \quad a^2 + bdc + dcb + b^3 = 0.
\end{align*}
\]
Full resolved space given by moduli space of D0 ( \( \vec{d} = (1, 2) \) rep):

\[
\begin{align*}
C & \quad \text{with relations} \\
\mathbb{C} & \quad (b^2 + dc)d = 0, \quad c(b^2 + dc) = 0, \quad ab + ba = 0, \quad a^2 + bdc + dc + b^3 = 0.
\end{align*}
\]

Resolved space given by
- all possible values of maps \( a, b, c, d \) (except ‘SR-ideal’),
- subject to realtions and
- modded by gauge transformations \( U(1) \times U(2) \) and D-terms:

\[
\begin{align*}
d^\dagger d - cc^\dagger &= 2\theta \\
dd^\dagger - c^\dagger c + [a, a^\dagger] + [b, b^\dagger] &= \theta 1_2
\end{align*}
\]
Exceptional $\mathbb{P}^1$: locus that is pushed down to $(y, s, w, z)$ in sing space.

For both phases:

$$a^2 = 0, \quad b^2 = 0, \quad ab + ba = 0$$

and

$$c = 0 \text{ for } \theta > 0, \quad \text{while} \quad d = 0 \text{ for } \theta < 0$$

Interpolate between the two phases by taking

$$a = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} \sqrt{2} \gamma & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ \sqrt{2} \delta \end{pmatrix}$$

D-terms become

$$|\delta|^2 - |\gamma|^2 = \theta, \quad |\alpha|^2 + |\beta|^2 = |\delta|^2 + |\gamma|^2$$

Now, take e.g. phase $\theta > 0$. Then $\gamma = 0$ and $\mathbb{P}^1$ spanned by $\alpha, \beta$ with residual gauge transformation $(\alpha, \beta) \mapsto (\lambda \alpha, \lambda \beta)$.

Changing continuously $\theta$, follow the flop transition through the singularity.
Consider phase $\theta > 0$.

Extra divisor given by locus where two sections of $M$ become parallel:

$$\sigma_1(ad \wedge d) + \sigma_2(bd \wedge d) + \sigma_3(abd \wedge d) = 0.$$  

( Sections of $M$ are $d, ad, bd, abd$. )

- Again family of divisors intersecting the $\mathbb{P}^1$ at one point: $\sigma_1\alpha + \sigma_2\beta = 0$.
- It corresponds to what one finds working in the sing space:

$$
\begin{pmatrix}
-s w^2 - z^2 & -y w + s z & -w s^2 - x z \\
y w + z s & -w^3 - s^2 & -z w^2 + y s \\
s^2 w - y z & z w^2 + y s & -s w^3 - y^2
\end{pmatrix} \cdot 
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{pmatrix} = 0
$$

( Before we specialized $\vec{\sigma} = (1, 0, 0)$. )
Fractional branes are $D0_R$ with $\vec{d} = (0, 1)$ and $D0_L$ with $\vec{d} = (1, 0)$.

Mobile D0 splits into $D0_L \oplus 2 \, D0_R$.

- $D0_R$ has D2-charge equal to 1.
- $D0_R$ has D2-charge equal to 2.

Charge-1 fractional brane gives **massless charged state** in the singular limit. 
[Aspinwall,Morrison]

If it worked like in the conifold, changing the B field the charge-2 state may become massless. ( Possibility to have double charge states. )
Conclusions

- Matrix factorization and quiver for fourfolds with massless U(1).
- Naturally encode extra U(1) divisor (already in the singular limit): family of representatives that includes extra section of elliptic fibration.
- Associated quiver gives resolution. Exceptional $\mathbb{P}^1$. Flop transition.
- Massless matter as fractional branes (quiver).

Open questions:

- More complicated geometries.
- Higher charge states?
- Results presented here for three-fold. Four-fold at the edge of math research.

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The End