Moduli of Heterotic Backgrounds and their Effective Theories

Eirik Eik Svanes (ILP, LPTHE)
Based on work in collaboration with
A. Ashmore, X. de la Ossa, E. Hardy, M. Larfors,
R. Minasian, C. Strickland-Constable.
arXiv:1402.1725, 1409.3347, 1509.08724,
1607.03473, 1704.08717, 17xx.xxxx, etc.

July 4th 2017, String Phenomenology 2017, Virginia Tech
Introduction
The low energy theory of the heterotic string is ten dimensional supergravity coupled to Yang-Mills gauge theory.

Easy to obtain four-dimensional supersymmetric grand unified theories from compactifications on Calabi-Yau manifolds [Candelas et al 85, ..].
The low energy theory of the heterotic string is ten dimensional supergravity coupled to Yang-Mills gauge theory.

Easy to obtain four-dimensional supersymmetric grand unified theories from compactifications on Calabi-Yau manifolds [Candelas et al 85, ..].

Complications:

- Higher curvature corrections induce torsional (non Ricci-flat) geometries.
- Harder to understand geometries. Loose toolbox of algebraic geometry and Kähler geometry.
- Harder to understand moduli (the deformation space).
The low energy theory of the heterotic string is ten dimensional supergravity coupled to Yang-Mills gauge theory.

Easy to obtain four-dimensional supersymmetric grand unified theories from compactifications on Calabi-Yau manifolds [Candelas et al 85, ..].

Complications:

- Higher curvature corrections induce torsional (non Ricci-flat) geometries.
- Harder to understand geometries. Loose toolbox of algebraic geometry and Kähler geometry.
- Harder to understand moduli (the deformation space).

This talk: Heterotic string on manifolds with reduced structure group $SU(3)$ (and $G_2$ manifolds).

- Finite deformations of $SU(3)$ system, the superpotential and “effective theory” governing heterotic DGLA.
- (Short) review of infinitesimal (massless) moduli of $SU(3)$ compactifications.
- Higher order deformations and obstructions.
- (Infinitesimal moduli of heterotic string on $G_2$-structure Manifolds.)
3 Steps in understanding moduli of a stringy geometry:
3 Steps in understanding moduli of a stringy geometry:

- **Step 1:** Infinitesimal *massless* spectrum $T\mathcal{M}$. Derive differential $\mathcal{D}$ with $\mathcal{D}^2 = 0$. Tangent space of moduli space then given by cohomology

$$T\mathcal{M} = H^1_{\mathcal{D}}(\mathcal{Q}).$$

Moduli fields $\mathcal{X}$ usually one-forms with values in a bundle $\mathcal{Q}$ (or sheaf), naturally associated to the given moduli problem.
3 Steps in understanding moduli of a stringy geometry:

- **Step 1**: Infinitesimal *massless* spectrum $\mathcal{TM}$. Derive differential $\mathcal{D}$ with $\mathcal{D}^2 = 0$. Tangent space of moduli space then given by cohomology

$$\mathcal{TM} = H_{\mathcal{D}}^1(\mathcal{Q}) .$$

Moduli fields $\mathcal{X}$ usually one-forms with values in a bundle $\mathcal{Q}$ (or sheaf), naturally associated to the given moduli problem.

- **Step 2**: Understand geometry of $\mathcal{M}$ (Kähler metric, etc). Higher order deformations, obstructions (Yukawa couplings). Maurer-Cartan elements,

$$\mathcal{D}\mathcal{X} + \frac{1}{2}[\mathcal{X}, \mathcal{X}] = 0 ,$$

and associated differentially graded Lie algebra (or $L_\infty$-algebra).
3 Steps in understanding moduli of a stringy geometry:

- **Step 1:** Infinitesimal *massless* spectrum $T\mathcal{M}$. Derive differential $\mathcal{D}$ with $\mathcal{D}^2 = 0$. Tangent space of moduli space then given by cohomology

$$T\mathcal{M} = H^1_D(\mathcal{Q}).$$

Moduli fields $\mathcal{X}$ usually one-forms with values in a bundle $\mathcal{Q}$ (or sheaf), naturally associated to the given moduli problem.

- **Step 2:** Understand geometry of $\mathcal{M}$ (Kähler metric, etc). Higher order deformations, obstructions (Yukawa couplings). Maurer-Cartan elements,

$$\mathcal{D}\mathcal{X} + \frac{1}{2}[\mathcal{X}, \mathcal{X}] = 0,$$

and associated differentially graded Lie algebra (or $L_\infty$-algebra).

- **Step 3:** Understand *quantum cohomology ring*. Include non perturbative effects such as world-sheet instantons, and quantum corrections (higher genus effects). Construct topological theory of corresponding structures?
The Heterotic $SU(3)$-system

\[ \mathcal{M}_{10} = \mathcal{M}_4 \times X_6 \]
Compactification on six dimensional compact $\text{SU}(3)$-structure manifold $(X, \Omega, \omega)$ results in a 4d $N = 1$ supergravity coupled to Yang-Mills.
Compactification on six dimensional compact $SU(3)$-structure manifold $(X, \Omega, \omega)$ results in a 4d $\mathcal{N} = 1$ supergravity coupled to Yang-Mills. This theory has a superpotential given by [Becker et al 03, Cardoso et al 03, Lukas et al 05, McOrist 16, ..]

$$W = \int_X (H + i d\omega) \wedge \Omega,$$

where the flux is given by

$$H = dB + \frac{\alpha'}{4} \omega_{CS}(A).$$

Note that the flux $H$ is gauge invariant which imposes a gauge transformation on $B$ through the Green-Schwarz mechanism.
Compactification on six dimensional compact $SU(3)$-structure manifold $(X, \Omega, \omega)$ results in a 4d $N = 1$ supergravity coupled to Yang-Mills. This theory has a superpotential given by [Becker et al 03, Cardoso et al 03, Lukas et al 05, McOrist 16, ..]

$$W = \int_X (H + i\omega) \wedge \Omega,$$

where the flux is given by

$$H = dB + \frac{\alpha'}{4} \omega_{CS}(A).$$

Note that the flux $H$ is gauge invariant which imposes a gauge transformation on $B$ through the Green-Schwarz mechanism.

Supersymmetric (Minkowski) solutions require that:

$$\delta W = W = 0,$$

at the solution.
From the superpotential we derive the following conditions on the $SU(3)$ structure [Strominger, Hull 86]

- $d\Omega = 0 \Rightarrow (X, J)$ is a complex manifold. Equivalent to $\partial^2 = 0$, where
  \[ d = \partial + \overline{\partial} \]
  and
  \[ \overline{\partial} : \Omega^{(p,q)}(X) \to \Omega^{(p,q+1)}(X) \].
  The operator $\overline{\partial}$ defines Dolbeault cohomologies $H_{\overline{\partial}}^{(p,q)}(X)$.

- $F \wedge \Omega = 0 \iff F^{(0,2)} = \overline{\partial}_A^2 = 0$, where $\overline{\partial}_A = d_A^{(0,1)}$. We say the bundle $(V, A)$ is holomorphic.

- $H = i(\partial - \overline{\partial})\omega$ and the flux is identified with internal torsion.
From the superpotential we derive the following conditions on the $SU(3)$ structure [Strominger, Hull 86]

- $d\Omega = 0 \Rightarrow (X, J)$ is a complex manifold. Equivalent to $\overline{\partial}^2 = 0$, where $d = \partial + \overline{\partial}$ and $\overline{\partial} : \Omega^{(p,q)}(X) \to \Omega^{(p,q+1)}(X)$.

  The operator $\overline{\partial}$ defines Dolbeault cohomologies $H^{(p,q)}(X)$.

- $F \wedge \Omega = 0 \iff F^{(0,2)} = \overline{\partial}^2 A = 0$, where $\overline{\partial} A = d A^{(0,1)}$. We say the bundle $(V, A)$ is holomorphic.

- $H = i(\partial - \overline{\partial})\omega$ and the flux is identified with internal torsion.

There are also “D-term” conditions:

$$d \left( e^{-2\phi} \omega \wedge \omega \right) = 0 , \quad \omega \wedge \omega \wedge F = 0 ,$$

$X$ is conformally balanced and $A$ is Yang-Mills. This talk: Will assume stable bundles. D-terms not very relevant for moduli problem.
Considering *generic* higher deformations of the heterotic $SU(3)$-system is in general a very hard problem. Get some complicated $L_\infty$-algebra.
Considering *generic* higher deformations of the heterotic $SU(3)$-system is in general a very hard problem. Get some complicated $L_\infty$-algebra.

**Clues from physics:**

Know that superpotential is *holomorphic* $\Rightarrow$ A finite and *holomorphic* deformation of the heterotic $SU(3)$-system can be represented as a $(0, 1)$-form

$$y = (x, \alpha, \mu) \in \Omega^{(0,1)}(Q_2), \quad Q_2 = T^{*(1,0)}X \oplus \text{End}(V) \oplus T^{(1,0)}X,$$

where $\mu \in \Omega^{(0,1)}(T^{(1,0)}X)$, $\alpha \in \Omega^{(0,1)}(\text{End}(V))$ and $x \in \Omega^{(0,1)}(T^{*(1,0)}X)$. 
Considering *generic* higher deformations of the heterotic $SU(3)$-system is in general a very hard problem. Get some complicated $L_\infty$-algebra.

**Clues from physics:**

Know that superpotential is *holomorphic* $\Rightarrow$ A *finite* and *holomorphic* deformation of the heterotic $SU(3)$-system can be represented as a $(0, 1)$-form

$$y = (x, \alpha, \mu) \in \Omega^{(0,1)}(Q_2), \quad Q_2 = T^{(1,0)}X \oplus \text{End}(V) \oplus T^{(1,0)}X,$$

where $\mu \in \Omega^{(0,1)}(T^{(1,0)}X), \alpha \in \Omega^{(0,1)}(\text{End}(V))$ and $x \in \Omega^{(0,1)}(T^{\ast(1,0)}X).$

A generic holomorphic deformation of the superpotential gives

$$\Delta W = \int_X \left( \langle y, \overline{\partial}_2 y \rangle + \frac{1}{3} \langle y, [y, y] \rangle + \mu^a \partial a b \right) \wedge \Omega,$$

where for $y_1, y_2 \in \Omega^{(0,\ast)}(Q_2), \langle y_1, y_2 \rangle = \mu_1^a x_2 a + \mu_2^a x_1 a + \text{tr} (\alpha_1 \alpha_2), \quad \mu \in \Omega^{(0,1)}(T^{(1,0)}X), \alpha \in \Omega^{(0,1)}(\text{End}(V))$ and $x \in \Omega^{(0,1)}(T^{\ast(1,0)}X)$ is auxiliary, and

$$\left[ , \right] : \ \Omega^{(0,p)}(Q_2) \times \Omega^{(0,q)}(Q_2) \to \Omega^{(0,p+q)}(Q_2)$$

satisfies Leibniz rule w.r.t. $\overline{\partial}_2$, and Jacobi identity modulo $\partial_a$-exact terms.
The field \( b \) is an auxiliary field and part of the deformation of \( B \). Integrating it out gives the condition
\[
\partial \Omega(\mu) = 0 , \quad \Omega(\mu) = \frac{1}{2} \Omega_{abc} \mu^a \, dz^{bc} ,
\]
and the “effective action” now reads
\[
\Delta W = \int_X \left( \langle y, \bar{\partial} y \rangle + \frac{1}{3} \langle y, [y, y] \rangle \right) \wedge \Omega ,
\]
Couplings between hermitian and complex moduli essential \( \Rightarrow \) heterotic kähler and complex structure moduli are \textit{not independent} (In contrast to Great Britain and the United States).
The field $b$ is an auxiliary field and part of the deformation of $B$. Integrating it out gives the condition

$$\partial \Omega(\mu) = 0, \quad \Omega(\mu) = \frac{1}{2} \Omega_{abc} \mu^a dz^{bc},$$

and the “effective action” now reads

$$\Delta W = \int_X \left( \langle y, \bar{\partial} y \rangle + \frac{1}{3} \langle y, [y, y] \rangle \right) \wedge \Omega,$$

Couplings between hermitian and complex moduli essential $\Rightarrow$ heterotic kähler and complex structure moduli are not independent (In contrast to Great Britain and the United States).

The action is invariant under the symmetries

$$y \rightarrow y + \bar{\partial} c + [y, c] + \partial d, \quad c \in \Omega^{0}(Q_2), \quad d \in \Omega^{(0,1)}(X).$$

These symmetry transformations can be identified with diffeomorphisms, gauge transformations and $B$-field transformations of the background fields.
The field $b$ is an auxiliary field and part of the deformation of $B$. Integrating it out gives the condition
\[ \partial \Omega(\mu) = 0, \quad \Omega(\mu) = \frac{1}{2} \Omega_{abc} \mu^a \, d^b c, \]
and the “effective action” now reads
\[ \Delta W = \int_X \left( \langle y, \bar{\partial}^2 y \rangle + \frac{1}{3} \langle y, [y, y] \rangle \right) \wedge \Omega, \]
Couplings between hermitian and complex moduli essential $\Rightarrow$ heterotic kähler and complex structure moduli are *not independent* (In contrast to Great Britain and the United States).

The action is invariant under the symmetries
\[ y \rightarrow y + \bar{\partial}^2 c + [y, c] + \partial_a d, \quad c \in \Omega^0(Q_2), \quad d \in \Omega^{(0,1)}(X). \]
These symmetry transformations can be identified with diffeomorphisms, gauge transformations and $B$-field transformations of the background fields.

**Equation of Motion:**
\[ \bar{\partial}^2 y + \frac{1}{2} [y, y] = \partial_a\text{-exact}. \]

The MC-equation for the DGLA $\mathcal{A} = \left( \Omega^{(0,*)}(Q_2/\{\partial_a\text{-exact}\}) \right) \wedge \bar{\partial}^2, [\ , \ ]).$
Note: Will assume “technical assumption” such as the $\bar{\partial}\partial$-lemma, or $h^{(0,1)} = 0$. 
Note: Will assume “technical assumption” such as the $\partial \bar{\partial}$-lemma, or $h^{(0,1)} = 0$.

The infinitesimal (massless) moduli $y$ satisfy the equation

$$\bar{\partial}_2 y = \partial_\alpha \text{-exact},$$

where $\bar{\partial}_2 = \bar{\partial}_1 + \mathcal{H}$ defines $Q_2$ as an extension [Anderson et al 14]

$$[\mathcal{H}] \in \text{Ext}^1(Q_1, T^*(1,0)X) = H_{\bar{\partial}_1}^{(0,1)}(Q_1^*, T^*(1,0)X).$$

Here $Q_1$ is the extension of Atiyah defined by $\bar{\partial}_1 = \bar{\partial} + \mathcal{F}$ (curvature)

$$[\mathcal{F}] \in \text{Ext}^1(T^{(1,0)}X, \text{End}(V)) = H_{\bar{\partial}}^{(0,1)}(T^*(1,0)X, \text{End}(V)).$$
Note: Will assume “technical assumption” such as the $\partial\overline{\partial}$-lemma, or $h^{(0,1)} = 0$.

The infinitesimal (massless) moduli $y$ satisfy the equation

$$\overline{\partial}^2 y = \partial_\alpha\text{-exact},$$

where $\overline{\partial}^2 = \overline{\partial}_1 + \mathcal{H}$ defines $Q_2$ as an extension [Anderson et al 14]

$$[\mathcal{H}] \in \text{Ext}^1(Q_1, T^{(1,0)}X) = H^{(0,1)}_{\overline{\partial}_1}(Q_1^*, T^{(1,0)}X).$$

Here $Q_1$ is the extension of Atiyah defined by $\overline{\partial}_1 = \overline{\partial} + \mathcal{F}$ (curvature)

$$[\mathcal{F}] \in \text{Ext}^1(T^{(1,0)}X, \text{End}(V)) = H^{(0,1)}_{\overline{\partial}}(T^{*(1,0)}X, \text{End}(V)).$$

Note: $\overline{\partial}^2 = 0 \iff (X, \Omega, \omega)$ is complex with a holomorphic vector bundle $(\text{End}(V), A)$ and satisfies heterotic Bianchi Identity. Applying “technical assumption”, and modding out by symmetries one finds

$$[y] \in H^{(0,1)}_{\overline{\partial}^2}(Q_2) = H^{(0,1)}_{\overline{\partial}}(T^{*(1,0)}X) \oplus \ker(\mathcal{H})$$

$$\ker(\mathcal{H}) \subseteq H^{(0,1)}_{\overline{\partial}_1}(Q_1) = H^{(0,1)}_{\overline{\partial}}(\text{End}(V)) \oplus \ker(\mathcal{F}),$$

as was shown in [Anderson et al 14, delaOssa-EES 14, Garcia-Fernandez 15].
The classically unobstructed fields $y$ satisfy the Maurer-Cartan equation

$$
\overline{\partial}^2 y + \frac{1}{2} [y, y] = \partial^a - \text{exact}
$$

The moduli satisfying this equation correspond to free fields without couplings in the effective 4d Lagrangian.
The classically unobstructed fields $y$ satisfy the Maurer-Cartan equation

$$\overline{\partial}_2 y + \frac{1}{2} [y, y] = \partial_a \text{-exact}$$

The moduli satisfying this equation correspond to free fields without couplings in the effective 4d Lagrangian.

Hard to analyse the obstructions in general, but can make observations for special cases such as for “Calabi-Yau” compactifications.

Theorem [Anderson-Gray-Lukas-Ovrut 11]: The number of unobstructed moduli of the Atiyah algebroid $Q_1$ on a Calabi-Yau is bounded from below by $h^{(0,1)}(T^{(1,0)}X)$. 
The classically unobstructed fields \( y \) satisfy the Maurer-Cartan equation

\[
\overline{\partial}_2 y + \frac{1}{2} [y, y] = \partial_a \text{-exact}
\]

The moduli satisfying this equation correspond to free fields without couplings in the effective 4d Lagrangian.

Hard to analyse the obstructions in general, but can make observations for special cases such as for “Calabi-Yau” compactifications.

Theorem [Anderson-Gray-Lukas-Ovrut 11]: The number of unobstructed moduli of the Atiyah algebroid \( Q_1 \) on a Calabi-Yau is bounded from below by \( h^{(0,1)}(T^{(1,0)}X) \).

Including \( \alpha' \) effects and the heterotic anomaly cancellation condition (Bianchi Identity), this can be extended to

Theorem [delaOssa-Hardy-EES 15]: The number of unobstructed moduli of the heterotic algebroid \( Q_2 \) on a “Calabi-Yau” is bounded from below by the number of massless geometric moduli of the base \( X \).
Conclusions and Outlook
Conclusions:

- Heterotic compactifications give a fertile ground for phenomenology, but the moduli problem is hard.

- From the heterotic superpotential we have derived an effective theory whose equation of motion naturally gives the MC equation of the heterotic DGLA.

- We have (re-)derived the infinitesimal moduli of the heterotic $SU(3)$-system, and considered higher order deformations and obstructions.
Conclusions and Outlook

Conclusions:

■ Heterotic compactifications give a fertile ground for phenomenology, but the moduli problem is hard.

■ From the heterotic superpotential we have derived an effective theory whose equation of motion naturally gives the MC equation of the heterotic DGLA.

■ We have (re-)derived the infinitesimal moduli of the heterotic $SU(3)$-system, and considered higher order deformations and obstructions.

Outlook, and work in progress:

■ Apply to explicit examples to connect with phenomenology, other areas of string theory (AdS/CFT?), and differential geometry?

■ What about non-perturbative effects, world sheet instantons, NS5-branes? Correct the Bianchi Identity and spoils holomorphic structure $\overline{\partial}^2 \neq 0$.

■ Quantise quasi-topological action $\Delta W$? Is there a corresponding topological world-sheet theory (e.g. AKSZ model or holomorphic beta-gamma system)? Compute invariants for heterotic geometries generalising DT-invariants and GW-invariants (relate to threshold corrections?). In progress with A. Ashmore, X. de la Ossa R. Minasian, C. Strickland-Constable.
Thank you for your attention, and happy 4th of July!
Compactifications to 3d

\[ \mathcal{M}_{10} = \mathcal{M}_3 \times Y_7 \]
Supersymmetry requires the internal geometry $Y$ to have a $G_2$-structure. Tangent bundle has a reduced structure group $G_2 \subset SO(7)$.

$G_2$ structure determined by a non-degenerate $G_2$ invariant three-form $\varphi$, where $\psi = *\varphi$ and the Hodge dual is given by the metric defined by $\varphi$.

Torsion classes (decomposed into irreducible representations of $G_2$)

$$d\varphi = \tau_1 \psi + 3 \tau_7 \wedge \varphi + *\tau_{27}$$
$$d\psi = 4 \tau_7 \wedge \psi + *\tau_{14}.$$ 

Note that $d\varphi = d\phi = 0 \iff \nabla^{LC} \varphi = 0 \ (G_2 \text{ holonomy}).$
Supersymmetry requires the internal geometry $Y$ to have a $G_2$-structure. Tangent bundle has a reduced structure group $G_2 \subset SO(7)$.

$G_2$ structure determined by a non-degenerate $G_2$ invariant three-form $\varphi$, where $\psi = *\varphi$ and the Hodge dual is given by the metric defined by $\varphi$.

Torsion classes (decomposed into irreducible representations of $G_2$)

\[
d\varphi = \tau_1 \psi + 3 \tau_7 \wedge \varphi + *\tau_{27}
d\psi = 4 \tau_7 \wedge \psi + *\tau_{14}.
\]

Note that $d\varphi = d\phi = 0 \iff \nabla^L C \varphi = 0$ ($G_2$ holonomy).

Supersymmetry requires [Papadopoulos et al 05, Lukas et al 10, Gray et al 12, ..]

\[
\tau_1 = d\text{vol}_{M_3} H, \quad \tau_7 = \frac{1}{2} d\phi
\]

\[
\tau_{14} = 0, \quad \tau_{27} = -H + \frac{1}{6} \tau_1 \varphi - \tau_7 \psi.
\]

Note, this is an integrable $G_2$-structure.

Gauge bundle: $F \wedge \psi = 0 \iff$ Curvature is an instanton.
For integrable $G_2$ structure (in particular $G_2$ holonomy) we have the complex
[Reyes-Carrion 93, Fernandez et al 98]

$$0 \to \Omega^0_1 \xrightarrow{\dd} \Omega^1_7 \xrightarrow{\dd} \Omega^2_7 \xrightarrow{\dd} \Omega^3_1 \to 0.$$ 

Here $\dd = \pi \circ d$, where $\pi$ is the projection onto the appropriate $G_2$ representation.

This is a differential complex, i.e. $\dd^2 = 0$, iff the $G_2$ structure is integrable (in close analogy with integrable complex structure $\iff \overline{\partial}^2 = 0$).
For integrable $G_2$ structure (in particular $G_2$ holonomy) we have the complex [Reyes-Carrion 93, Fernandez et al 98]

$$0 \to \Omega^0_1 \xrightarrow{\tilde{d}} \Omega^1_7 \xrightarrow{\tilde{d}} \Omega^2_7 \xrightarrow{\tilde{d}} \Omega^3_1 \to 0.$$  

Here $\tilde{d} = \pi \circ d$, where $\pi$ is the projection onto the appropriate $G_2$ representation.

This is a differential complex, i.e. $\tilde{d}^2 = 0$, iff the $G_2$ structure is integrable (in close analogy with integrable complex structure $\Leftrightarrow \bar{\partial}^2 = 0$).

This complex generalises to bundles $(V, A)$

$$0 \to \Omega^0_1(V) \xrightarrow{\tilde{d}_A} \Omega^1_7(V) \xrightarrow{\tilde{d}_A} \Omega^2_7(V) \xrightarrow{\tilde{d}_A} \Omega^3_1(V) \to 0.$$  

where $\tilde{d}_A = \pi \circ d_A$. This is a differential complex iff $F \wedge \psi = 0$.

These complexes are elliptic [Reyes-Carrion 93] $\Rightarrow$ the corresponding cohomologies $\tilde{H}^*(V)$ are finite dimensional.
The heterotic $G_2$ system naturally gives rise to a differential

$$
\mathcal{D} = \left( \begin{array}{cc}
\tilde{d}_A & \tilde{F} \\
\tilde{F} & \tilde{d}_\theta
\end{array} \right) : \tilde{\Omega}^p \left( \frac{\text{End}(V)}{TY} \right) \rightarrow \tilde{\Omega}^{p+1} \left( \frac{\text{End}(V)}{TY} \right),
$$

where the map(s) $\tilde{F}$ are given by

$$
\Delta \in \Omega^p(TY) : \tilde{F}(\Delta) = \pi \left( F_{mn} dx^n \wedge \Delta^m \right)
$$

$$
\alpha \in \Omega^p(\text{End}(V)) : \tilde{F}(\alpha)^m = \frac{\alpha'}{4} \pi \left[ \text{tr} \left( g^{mn} F_{nq} dx^q \wedge \alpha \right) \right].
$$

and $\pi$ denotes the appropriate projection.
The heterotic $G_2$ system naturally gives rise to a differential

$$\tilde{\mathcal{D}} = \left( \begin{array}{cc} \tilde{d}_A & \tilde{\mathcal{F}} \\ \tilde{\mathcal{F}} & \tilde{d}_\theta \end{array} \right) : \tilde{\Omega}^p \left( \begin{array}{c} \text{End}(V) \\ TY \end{array} \right) \rightarrow \tilde{\Omega}^{p+1} \left( \begin{array}{c} \text{End}(V) \\ TY \end{array} \right),$$

where the map(s) $\tilde{\mathcal{F}}$ are given by

$$\Delta \in \Omega^p(TY) : \tilde{\mathcal{F}}(\Delta) = \pi \left( F_{mn} dx^n \wedge \Delta^m \right)$$

$$\alpha \in \Omega^p(\text{End}(V)) : \tilde{\mathcal{F}}(\alpha)^m = \frac{\alpha^t}{4} \pi \left[ \text{tr} \left( g^{mn} F_{nq} dx^q \wedge \alpha \right) \right].$$

and $\pi$ denotes the appropriate projection.

The connection $d_\theta$ has connection symbols

$$\theta_{mn}^p = \Gamma_{nm}^p,$$

where $\Gamma$ denotes the unique metric connection with totally anti-symmetric torsion preserving the $G_2$ structure, $\nabla^\Gamma \varphi = 0$.

Using the heterotic supersymmetry conditions, it can be shown that $\tilde{\mathcal{D}}^2 = 0$. 
Theorem (delaOssa-Larfors-EES) The infinitesimal moduli space of the heterotic $G_2$ system is

$$TM = \tilde{H}^1_D(Q_1).$$

Scetch of proof: Identify infinitesimal deformations with $\tilde{D}$-closed forms. Identify heterotic symmetries (diffeomorphisms and gauge-transformations, $B$-field transformations) with $\tilde{D}$-exact forms.
Theorem (delaOssa-Larfors-EES) The infinitesimal moduli space of the heterotic $G_2$ system is

$$T\mathcal{M} = \tilde{H}_D^1(Q_1).$$

Sketch of proof: Identify infinitesimal deformations with $\tilde{\mathcal{D}}$-closed forms. Identify heterotic symmetries (diffeomorphisms and gauge-transformations, $B$-field transformations) with $\tilde{\mathcal{D}}$-exact forms.

Note when $\alpha' = 0$ the differential reads

$$\tilde{\mathcal{D}} = \begin{pmatrix} \tilde{d}_A & \tilde{F} \\ 0 & \tilde{d}_\theta \end{pmatrix} : \tilde{\Omega}^p \begin{pmatrix} \text{End}(V) \\ T\mathcal{Y} \end{pmatrix} \rightarrow \tilde{\Omega}^{p+1} \begin{pmatrix} \text{End}(V) \\ T\mathcal{Y} \end{pmatrix}.$$

Get short exact extension of complexes, $Q_1 = \text{End}(V) \oplus T\mathcal{Y}$

$$0 \rightarrow \tilde{\Omega}^*(\text{End}(V)) \rightarrow \tilde{\Omega}^*(Q_1) \rightarrow \tilde{\Omega}^*(T\mathcal{Y}) \rightarrow 0.$$

Long exact sequence in cohomology gives

$$T\mathcal{M}_1 = \tilde{H}^1(Q_1) = \tilde{H}^1(\text{End}(V)) \oplus \ker(\tilde{F}), \quad \ker(\tilde{F}) \subseteq \tilde{H}^1(T\mathcal{Y}),$$

where $\tilde{H}^1(T\mathcal{Y})$ contains both metric and $B$-field deformations.
The heterotic $G_2$-system has advantages over the $SU(3)$-system:
The heterotic $G_2$-system has advantages over the $SU(3)$-system:

- The system is more compact (not an extension).
- No separation of structures, $\tilde{D}$ has everything. Opposite direction holds within perturbative curvature expansion: $\tilde{D}^2 = 0$ implies most of the heterotic $G_2$-system, except that Lee form $\tau_7$ is exact (given by the dilaton).
- Embedding $SU(3)$-system in $G_2$-system puts holomorphic and hermitian structures on equal footing.
- The differential $\tilde{D}$ makes it clear that the heterotic $SU(3)$-system can be viewed as a holomorphic Yang-Mills connection on $\mathcal{Q}$.
- No “technical assumption”.
The heterotic $G_2$-system has advantages over the $SU(3)$-system:

- The system is more compact (not an extension).
- No separation of structures, $\tilde{\mathcal{D}}$ has everything. Opposite direction holds within perturbative curvature expansion: $\tilde{\mathcal{D}}^2 = 0$ implies most of the heterotic $G_2$-system, except that Lee form $\tau_7$ is exact (given by the dilaton).
- Embedding $SU(3)$-system in $G_2$-system puts holomorphic and hermitian structures on equal footing.
- The differential $\tilde{\mathcal{D}}$ makes it clear that the heterotic $SU(3)$-system can be viewed as a holomorphic Yang-Mills connection on $Q$.
- No “technical assumption”.

The disadvantages are:

- The system is more compact (not an extension) $\Rightarrow$ harder to identify usual cohomologies within $H^1_{\tilde{\mathcal{D}}}(Q_1)$.
- Harder to compute $H^1_{\tilde{\mathcal{D}}}(Q_1)$. No clear notion of Poincare lemma or sheaf cohomology $\Rightarrow$ hard to get to by algebraic methods. Homogeneous examples?