LARGE FIELDS AND LATTICES

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based on work with

Ben Heidenreich, Cody Long, Liam McAllister, Matt Reece, and Tom Rudelius

String Pheno
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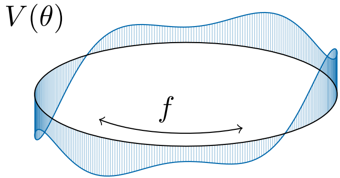
Motivation

Does quantum gravity permit controlled, super-Planckian field displacements?

Axions provide a particularly bright lamppost to study under.



Axion field space is $\mathbb{R}^N/\Gamma$, so the Lagrangian can be written as

$$\mathcal{L} = -\frac{f^2}{2}\delta_{ij}\partial_\mu\theta^i\,\partial^\mu\theta^j - \sum_{\mathbf{q}\in\Gamma} F_{\mathbf{q}}\,e^{i\delta_{\mathbf{q}}}\exp\left(2\pi i\mathbf{q}\cdot\boldsymbol{\theta}\right) + \ldots$$

[cf. talks by Blumenhagen, Flauger, Hebecker, Heidenreich, Herraez, Klaewer, McAllister, Montero, Ooguri, Rudelius, Shiu, Staessens, Valenzuela, Zoccarato]

Motivation

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This potential is generated non-perturbatively, and (s)LWGC demands

$$F_{\mathbf{q}} \propto \exp\left(-\pi\mu|\mathbf{q}|\right) \qquad \text{with} \qquad \mu \sim M_{\text{pl}}/f.$$

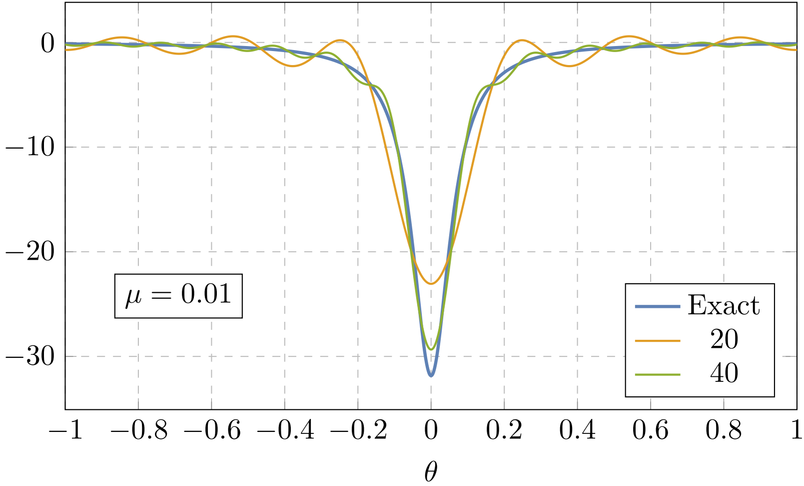
Fundamental period is super-Planckian at small μ,
but many terms are important!

Do these potentials admit large field ranges?

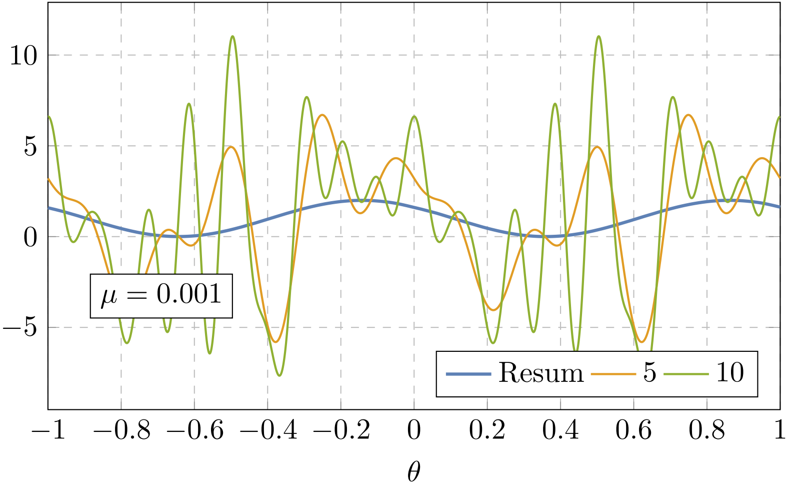
How do we understand the structure of these lattice sums?

[Heidenreich, Reece, Rudelius '16; Montero, Shiu, Soler '16]

$$V(\theta) = -\sum_{q\in\mathbb{Z}} e^{-2\pi\mu|q|+iq\theta}$$



$$V(\theta) = \sum_{q \in \mathbb{Z}} e^{-2\pi\mu|q| + iq\theta + iq^{33}}$$



Harmonic Variance

In this talk, we will study

$$V(\boldsymbol{\theta}) = \sum_{\mathbf{q}\in\Gamma} F_{\mathbf{q}} e^{i\delta_{\mathbf{q}}} \exp\left(2\pi i \mathbf{q}\cdot\boldsymbol{\theta}\right) \qquad \langle F_{\mathbf{q}}^2\rangle = e^{-2\pi\mu|\mathbf{q}|} \qquad \delta_{\mathbf{q}} \in [0, 2\pi)$$

Useful measure is the variance in the n-th harmonic along the $\bar{\mathbf{e}}$ direction,

$$\sigma_n^2 = \sum_{\mathbf{q}\in\Gamma}^{\mathbf{q}\cdot\bar{\mathbf{e}}=n} e^{-2\pi\mu|\mathbf{q}|} \quad \to \quad V(\theta\,\bar{\mathbf{e}}) \sim \sum_{n=0}^{\infty} \mathcal{N}(0, \sigma_n^2) \times \cos\left(2\pi n\theta\right)$$

How do we investigate small μ?

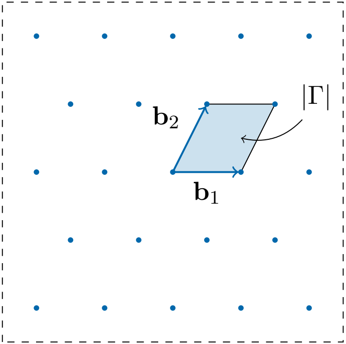
Can we find a better representation of these sums?

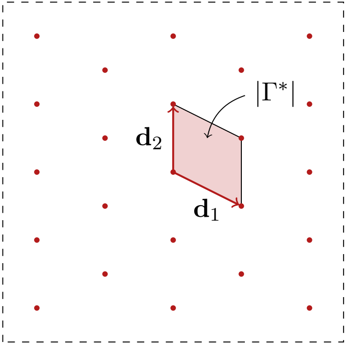
[Heidenreich, Long, McAllister, Reece, Rudelius, JS]

Poisson Resummation

We can use Poisson resummation to improve convergence.

$$\sum_{\mathbf{q}\in\Gamma} f(\mathbf{q}) = \sum_{\bar{\mathbf{q}}\in\Gamma^*} \hat{f}(\mathbf{q})$$

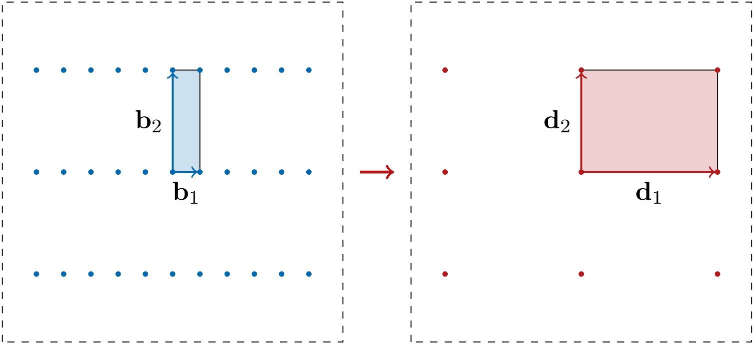




Sublattice Resummation

Can resum the entire lattice, or sublattices.

$$\sum_{\mathbf{q}\in\Gamma} f(\mathbf{q}) = \sum_{\bar{\mathbf{q}}\in\gamma} \sum_{\mathbf{q}\in\Gamma/\gamma} \hat{f}_\gamma(\mathbf{q};\bar{\mathbf{q}})$$



Which sublattice resummation yields the most truncatable sum?

Warmup

Can rewrite our sum to remove the constraint,

$$\sigma_n^2 = \sum_{\mathbf{q}\in\Gamma/\mathbf{e}} e^{-2\pi\mu|\mathbf{q}+n\mathbf{e}|}$$

This particular sum is complicated—not obvious when resummation helps.

As a warmup, consider the **Siegel theta function**,

$$\Theta_{\mathbf{pq}}\left(\mathbf{z}|\mathbf{\Omega}\right) = \sum_{\mathbf{k}\in\mathbb{Z}^N} e^{\pi i(\mathbf{k}+\mathbf{p})^\top\mathbf{\Omega}(\mathbf{k}+\mathbf{p})+2\pi i(\mathbf{k}+\mathbf{p})\cdot(\mathbf{z}+\mathbf{q})}$$

Riemann matrix, $\mathbf{\Omega}$, encodes the structure of lattice.

$$\Theta_{\mathbf{00}}\left(\mathbf{0}|i\mu\mathbf{Y}_\Gamma\right) = \Theta\left(\mathbf{0}|i\mu\mathbf{Y}_\Gamma\right) = \sum_{\mathbf{q}\in\Gamma} e^{-\pi\mu|\mathbf{q}|^2}$$

Siegel's Fundamental Domain

Each Riemann matrix $\mathbf{\Omega}$ has a representative

$$\mathbf{\Omega}' = (\mathbf{A}\mathbf{\Omega} + \mathbf{B})(\mathbf{C}\mathbf{\Omega} + \mathbf{D})^{-1} \qquad \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \in \mathrm{Sp}(2N, \mathbb{Z})$$

with "maximal imaginary part," called $\mathbf{\Omega}$'s representative in the **Siegel fundamental domain**.

$$\Theta_{\tilde{\mathbf{p}}\tilde{\mathbf{q}}}(0, \mathbf{\Omega}') = k\sqrt{\det(\mathbf{C}\mathbf{\Omega} + \mathbf{D})}\Theta_{\mathbf{pq}}(0, \mathbf{\Omega})$$

In this language, the transformation $\mathbf{B} = -\mathbb{1}$ and $\mathbf{C} = \mathbb{1}$ corresponds to Poisson resummation of the entire lattice.

Optimal representation is solved by finding Siegel representative!

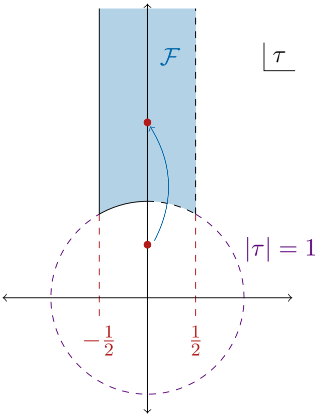
[cf. Frauendiener, Jaber, Klein '17; Deconinck, Heil, Bobenko, Hoeij, Schmies, '02]

One Dimension

In one dimension, this reduces to something very familiar.

Siegel theta reduces to a Jacobi theta, with lattice parameter $\tau = i\mu\mathbf{b}^2$.

Optimal truncation determined by τ's representative in $\mathrm{SL}(2,\mathbb{Z})$ fundamental domain.



$$\theta_3(0|i\mu\mathbf{Y}_\Gamma) = \begin{cases} \theta_3(0|i\mu\mathbf{b}^2) & \mu\mathbf{b}^2 \geq 1 \\ (\mu\mathbf{b}^2)^{-1/2}\,\theta_3(0|\frac{i}{\mu\mathbf{b}^2}) & \mu\mathbf{b}^2 \leq 1 \end{cases}$$

Summation to Siegel

Which sublattice do we resum?
Relate harmonic variance to Siegel's Θ!

$$e^{-2\pi\mu|\mathbf{q}|} = \mu \int_0^\infty \frac{\mathrm{d}t}{t^{3/2}}\, e^{-\pi\mu^2 t^{-1} - \pi \mathbf{q}^2 t}$$

$$\sigma_n^2 = \sum_{\mathbf{q}\in\Gamma/\mathbf{e}} e^{-2\pi\mu|\mathbf{q}+n\bar{\mathbf{e}}|}$$

$$\sigma_n^2 = \mu \int_0^\infty \mathrm{d}t\, \frac{1}{t^{3/2}} \exp\left[-\pi\left(\frac{\mu^2}{t} + n^2 \mathbf{e}_\perp^2\, t\right)\right] \Theta_{n\bar{\mathbf{e}}_\parallel \mathbf{0}}\left(\mathbf{0}|it\mathbf{Y}_{\Gamma/\mathbf{e}}\right)$$

$$\Theta_{\mathbf{pq}}\left(\mathbf{z}|\boldsymbol{\Omega}\right) = \sum_{\mathbf{k}\in\mathbb{Z}^N} e^{\pi i(\mathbf{k}+\mathbf{p})^\top \boldsymbol{\Omega}(\mathbf{k}+\mathbf{p}) + 2\pi i(\mathbf{k}+\mathbf{p})\cdot(\mathbf{z}+\mathbf{q})}$$

[Heidenreich, Long, McAllister, Reece, Rudelius, JS]

Summation to Siegel

This cleaves problem in two: summand and lattice.

Integrating over **kernel**, whose peak depends on μ.
Different summands correspond to different kernels.

$$\sigma_n^2 = \mu \int_0^\infty \mathrm{d}t \, \frac{1}{t^{3/2}} \exp\left[-\pi\left(\frac{\mu^2}{t} + n^2 \mathbf{e}_\perp^2\, t\right)\right] \Theta_{n\bar{\mathbf{e}}_\| \mathbf{0}}\left(\mathbf{0} \middle| it\mathbf{Y}_{\Gamma/\mathbf{e}}\right)$$

Riemann matrix encodes lattice structure,
and we integrate over all scalings of the lattice.

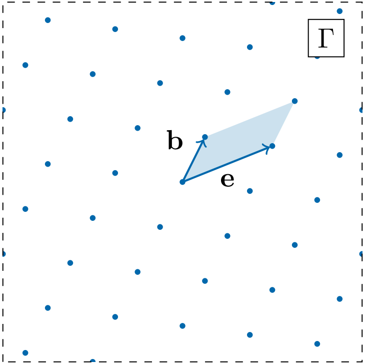
Integration interval is partitioned by representatives.

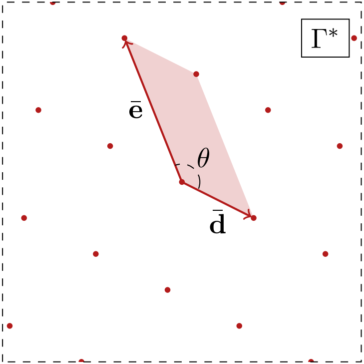
$$\sigma_n^2 = \sum_i \int_{\alpha_i} \mathrm{d}t\, \mathcal{K}_i(\mu, n, \mathbf{e}_\perp^2, t)\, \Theta_{\mathbf{p}_i\mathbf{q}_i}(\mathbf{0}|\boldsymbol{\Omega}_i(t))$$

[Heidenreich, Long, McAllister, Reece, Rudelius, JS]

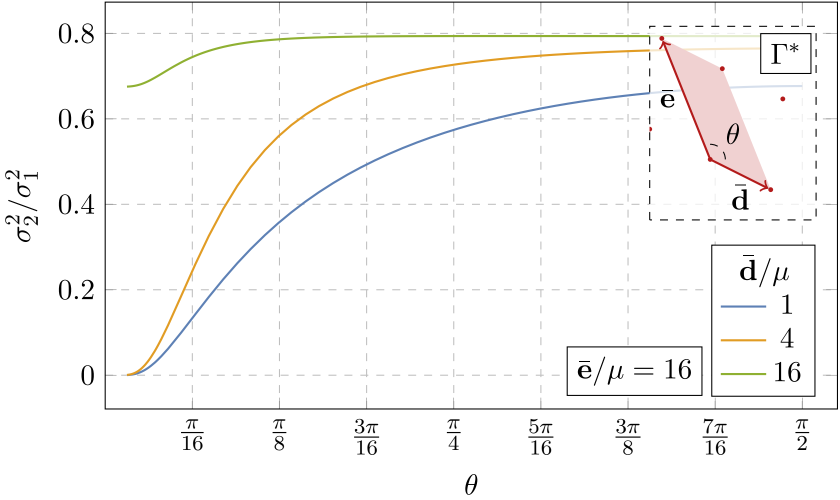
Two Fields

$$\sigma_n^2 = \frac{\mu}{|\mathbf{b}|}\int_0^{\mathbf{b}^{-2}} \frac{\mathrm{d}t}{t^2}\,\exp\left[-\pi\left(\frac{\mu^2}{t} + n^2\mathbf{e}_\perp^2\, t\right)\right]\Theta_{\mathbf{0},-n\bar{\mathbf{e}}_\parallel}\left(\mathbf{0}\middle|\frac{i}{t\mathbf{b}^2}\right)$$
$$+\,\mu\int_{\mathbf{b}^{-2}}^{\infty}\frac{\mathrm{d}t}{t^{3/2}}\,\exp\left[-\pi\left(\frac{\mu^2}{t} + n\mathbf{e}_\perp^2\, t\right)\right]\Theta_{n\bar{\mathbf{e}}_\parallel,\mathbf{0}}\left(\mathbf{0}\middle|it\mathbf{b}^2\right)$$





$$\sigma_n^2 = \sum_{\mathbf{q}\in\Gamma}^{\mathbf{q}\cdot\bar{\mathbf{e}}=n} e^{-2\pi\mu|\mathbf{q}|}$$



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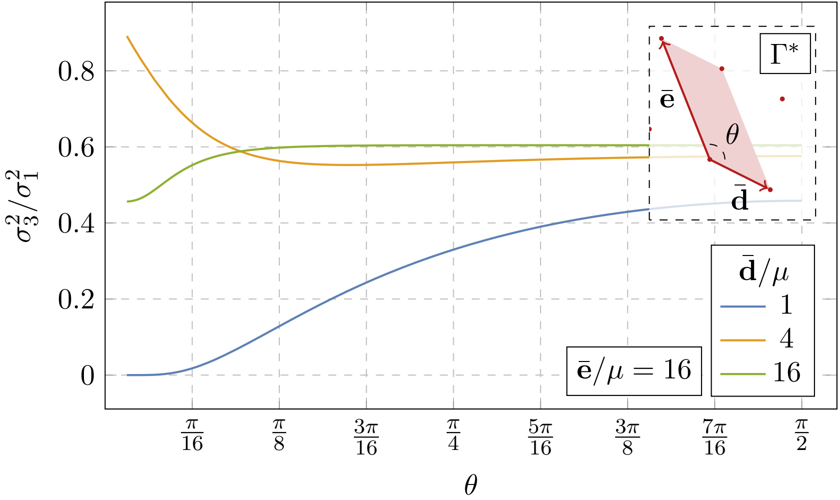

Conclusions

- Representative in Siegel's fundamental domain determines the optimal (most truncatable) representation of a lattice sum, and we may use this technology to study a large class of lattice sums.
- Can have suppressed higher harmonics at small μ (large f)!
- What structures are responsible for this suppression?
- What happens in higher-dimensional lattices?

Thanks!