LARGE FIELDS
AND
LATTICES

John Stout

based on work with

Ben Heidenreich, Cody Long, Liam McAllister, Matt Reece, and Tom Rudelius

String Pheno
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Motivation

Does quantum gravity permit controlled, super-Planckian field displacements?

Axions provide a particularly bright lamppost to study under.

Axion field space is \( \mathbb{R}^N / \Gamma \), so the Lagrangian can be written as

\[
\mathcal{L} = -f^2 \delta_{ij} \partial_\mu \theta^i \partial^\mu \theta^j - \sum_{\mathbf{q} \in \Gamma} F_\mathbf{q} e^{i \delta_\mathbf{q}} \exp (2\pi i \mathbf{q} \cdot \theta) + \ldots
\]

[cf. talks by Blumenhagen, Flauger, Hebecker, Heidenreich, Herraez, Klaewer, McAllister, Montero, Ooguri, Rudelius, Shiu, Staessens, Valenzuela, Zoccarato]
Motivation

\[ \mathcal{L} = -\frac{f^2}{2} \delta_{ij} \partial_\mu \theta^i \partial^\mu \theta^j - \sum_{\mathbf{q} \in \Gamma} F_\mathbf{q} \, e^{i\mathbf{q} \cdot \mathbf{\theta}} \exp \left( 2\pi i \mathbf{q} \cdot \mathbf{\theta} \right) + \ldots \]

This potential is generated non-perturbatively, and (s)LWGC demands

\[ F_\mathbf{q} \propto \exp \left( -\pi \mu |\mathbf{q}| \right) \quad \text{with} \quad \mu \sim M_{\text{pl}}/f. \]

Fundamental period is super-Planckian at small \( \mu \), but many terms are important!

**Do these potentials admit large field ranges?**

**How do we understand the structure of these lattice sums?**

[Heidenreich, Reece, Rudelius '16; Montero, Shiu, Soler '16]
\[ V(\theta) = -\sum_{q \in \mathbb{Z}} e^{-2\pi \mu |q| + iq\theta} \]
\[ V(\theta) = \sum_{q \in \mathbb{Z}} e^{-2\pi \mu |q|} + iq\theta + iq^{33} \]
Harmonic Variance

In this talk, we will study

\[ V(\theta) = \sum_{q \in \Gamma} F_q e^{i\delta_q} \exp (2\pi i q \cdot \theta) \quad \langle F_q^2 \rangle = e^{-2\pi |q|} \quad \delta_q \in [0, 2\pi) \]

Useful measure is the variance in the \( n \)-th harmonic along the \( \bar{e} \) direction,

\[ \sigma_n^2 = \sum_{q \in \Gamma, q \cdot \bar{e} = n} e^{-2\pi |q|} \rightarrow V(\theta \bar{e}) \sim \sum_{n=0}^{\infty} \mathcal{N}(0, \sigma_n^2) \times \cos (2\pi n \theta) \]

How do we investigate small \( \mu \)?

Can we find a better representation of these sums?

[Heidenreich, Long, McAllister, Reece, Rudelius, JS]
We can use Poisson resummation to improve convergence.

\[ \sum_{q \in \Gamma} f(q) = \sum_{\tilde{q} \in \Gamma^*} \hat{f}(q) \]
Can resum the entire lattice, or sublattices.

\[
\sum_{\mathbf{q} \in \Gamma} f(\mathbf{q}) = \sum_{\bar{\mathbf{q}} \in \gamma} \sum_{\mathbf{q} \in \Gamma / \gamma} \hat{f}_\gamma(\mathbf{q}; \bar{\mathbf{q}})
\]

Which sublattice resummation yields the most truncatable sum?
Warmup

Can rewrite our sum to remove the constraint,

\[ \sigma_n^2 = \sum_{q \in \Gamma/e} e^{-2\pi \mu|q + n\mathbf{e}|} \]

This particular sum is complicated—not obvious when resummation helps.

As a warmup, consider the **Siegel theta function**,

\[ \Theta_{pq}(\mathbf{z}|\Omega) = \sum_{\mathbf{k} \in \mathbb{Z}^N} e^{\pi i (\mathbf{k} + \mathbf{p})^\top \Omega (\mathbf{k} + \mathbf{p}) + 2\pi i (\mathbf{k} + \mathbf{p}) \cdot (\mathbf{z} + \mathbf{q})} \]

**Riemann matrix**, \( \Omega \), encodes the structure of lattice.

\[ \Theta_{00}(0|\mu \mathbf{Y}_{\Gamma}) = \Theta(0|\mu \mathbf{Y}_{\Gamma}) = \sum_{\mathbf{q} \in \Gamma} e^{-\pi \mu|\mathbf{q}|^2} \]
Siegel’s Fundamental Domain

Each Riemann matrix $\Omega$ has a representative

$$\Omega' = (A\Omega + B)(C\Omega + D)^{-1} \quad \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \text{Sp}(2N, \mathbb{Z})$$

with “maximal imaginary part,” called $\Omega$’s representative in the **Siegel fundamental domain**.

$$\Theta_{\tilde{p}\tilde{q}}(0, \Omega') = k \sqrt{\det(C\Omega + D)} \Theta_{pq}(0, \Omega)$$

In this language, the transformation $B = -1$ and $C = 1$ corresponds to Poisson resummation of the entire lattice.

Optimal representation is solved by finding Siegel representative!

[cf. Frauendiener, Jaber, Klein ’17; Deconinck, Heil, Bobenko, Hoeij, Schmies, ’02]
In one dimension, this reduces to something very familiar.

Siegel theta reduces to a Jacobi theta, with lattice parameter $\tau = i \mu b^2$.

Optimal truncation determined by $\tau$’s representative in $SL(2, \mathbb{Z})$ fundamental domain.

$$\theta_3(0| i \mu b^2) = \begin{cases} 
\theta_3(0| i \mu b^2) & \mu b^2 \geq 1 \\
(\mu b^2)^{-1/2} \theta_3(0| \frac{i}{\mu b^2}) & \mu b^2 \leq 1 
\end{cases}$$
Summation to Siegel

Which sublattice do we resum?
Relate harmonic variance to Siegel’s $\Theta$!

$$e^{-2\pi \mu |q|} = \mu \int_0^\infty \frac{dt}{t^{3/2}} e^{-\pi \mu^2 t^{-1} - \pi q^2 t}$$

$$\sigma_n^2 = \sum_{q \in \Gamma/e} e^{-2\pi \mu |q + n\bar{e}|}$$

$$\sigma_n^2 = \mu \int_0^\infty dt \frac{1}{t^{3/2}} \exp \left[ -\pi \left( \frac{\mu^2}{t} + n^2 e_\perp^2 t \right) \right] \Theta_{n\bar{e}\parallel 0} \left( 0 | it \mathbf{Y}_{\Gamma/e} \right)$$

$$\Theta_{pq} (z | \Omega) = \sum_{k \in \mathbb{Z}^N} e^{\pi i (k+p)^\top \Omega (k+p) + 2\pi i (k+p) \cdot (z+q)}$$

[Heidenreich, Long, McAllister, Reece, Rudelius, JS]
Summation to Siegel

This cleaves problem in two: summand and lattice.

Integrating over kernel, whose peak depends on $\mu$. Different summands correspond to different kernels.

$$\sigma_n^2 = \mu \int_0^\infty dt \frac{1}{t^{3/2}} \exp \left[ -\pi \left( \frac{\mu^2}{t} + n^2 e_\perp^2 t \right) \right] \Theta_{n \bar{\epsilon} || 0} (0 | itY_{\Gamma/e})$$

**Riemann matrix** encodes lattice structure, and we integrate over all scalings of the lattice.

Integration interval is partitioned by representatives.

$$\sigma_n^2 = \sum_i \int_{\alpha_i} dt \, K_i(\mu, n, e_\perp^2, t) \Theta_{p_i q_i} (0 | \Omega_i(t))$$

[Heidenreich, Long, McAllister, Reece, Rudelius, JS]
Two Fields

\[
\sigma_n^2 = \frac{\mu}{|b|} \int_0^{b^{-2}} \frac{dt}{t^2} \exp \left[-\pi \left( \frac{\mu^2}{t} + n^2 e_\perp^2 t \right) \right] \Theta_{0,-n\bar{e}\parallel} \left(0 \left| \frac{i}{tb^2} \right. \right) \\
+ \mu \int_{b^{-2}}^{\infty} \frac{dt}{t^{3/2}} \exp \left[-\pi \left( \frac{\mu^2}{t} + ne_\perp^2 t \right) \right] \Theta_{ne\parallel,0} \left(0 \left| itb^2 \right. \right)
\]

John Stout  jes554@cornell.edu
\[ \sigma_n^2 = \sum_{\mathbf{q} \in \Gamma} e^{-2\pi \mu |\mathbf{q}|} \]
\[ \sigma_n^2 = \sum_{q \in \Gamma} q \cdot \vec{e} = n e^{-2\pi \mu |q|} \]
Conclusions

- Representative in Siegel’s fundamental domain determines the optimal (most truncatable) representation of a lattice sum, and we may use this technology to study a large class of lattice sums.
- Can have suppressed higher harmonics at small $\mu$ (large $f$)?
- What structures are responsible for this suppression?
- What happens in higher-dimensional lattices?
Thanks!