Computational Algebraic Geometry meets String Theory
Rigid divisors and computing sheaf cohomology on Calabi-Yau hypersurfaces of toric 4-folds

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Calabi-Yau hypersurfaces X_3 of 4D toric varieties V_4

The setting for our talk is a Calabi-Yau hypersurface 3-fold in a 4D simplicial toric variety: $X \subset V$.

1. Start with a 4D reflexive polytope $\Delta \subset \mathbb{R}^4$, and its polar dual Δ^o. Place the single interior point at the origin.
2. Triangulate Δ^o (using fine, regular, star triangulation).
3. Define a simplicial fan Σ: rays are the rays from the origin to (other) lattice points of Δ^o. Cones are those induced by triangulation.
4. $V := \mathbb{P}_\Sigma$ is a simplicial toric variety, with at most point-like singularities.
5. Let F be a generic or random sum of monomials corresponding to lattice points of Δ.
6. Define $X := (F = 0) \subset V$. X is our smooth CY_3.

Important part to remember: the triangulation of Δ^o.

Rigid divisors and Hodge numbers of divisors on the CY3 X

Definition

If $D = D_2 \subset X$ is an effective divisor (i.e. a surface, possibly with several components),

- Let $h^i(\mathcal{O}_D) := \dim H^i(D, \mathcal{O}_D)$. The **Hodge vector** of D is

$$h^\bullet(\mathcal{O}_D) := (h^0(\mathcal{O}_D),\ h^1(\mathcal{O}_D),\ h^2(\mathcal{O}_D)).$$

- D is **rigid** if $h^\bullet(\mathcal{O}_D) = (1, 0, 0)$ (not necessarily smooth or irreducible!)

Computational problems of interest

- **Our main objective!** Given an effective divisor $D \subset X$, find $h^\bullet(\mathcal{O}_D)$.
- Given X, are there finitely many or infinitely many rigid divisors on X? (either case is interesting).
- Given X, computationally find all, or many, rigid divisors on X.

One goal for this talk

- Let Δ^o, $X \subset V = \mathbb{P}_\Sigma$ be as just constructed.
- Let the $\widehat{D}_i$ be the torus invariant prime divisors on V. These are in 1-1 correspondence with the (non-zero) lattice points of Δ^o.
- Let $D_i := \widehat{D}_i \cap X$. D_i is nonempty exactly when the lattice point lies on a 2-face of Δ^o, that is, is not internal to a facet of Δ^o.
- Let $D_1, \ldots, D_N$ be the collection of all of these divisors on X.

Key problem

Let D be a **square-free divisor**, i.e. $D = \sum_{i \in G} D_i$, where $G \subset \{1, \ldots, N\}$.

Goal: Compute the Hodge numbers

$$h^\bullet(\mathcal{O}_D) := \left(h^0(\mathcal{O}_D),\ h^1(\mathcal{O}_D),\ h^2(\mathcal{O}_D)\right).$$

Actually: will consider a somewhat larger class of divisors, which includes these.

Computing with these polyhedra and Calabi-Yau varieties using Macaulay2

- **Macaulay2**. An open source computer algebra system by Dan Grayson and myself, for investigations in algebraic geometry and related fields.
  1. Dan: one of the 7 original authors of Mathematica
  2. A community project: over $100 - 150$ user written packages in Macaulay2
- **StringTorics**. A new **Macaulay2** package, almost complete, designed to make the use of Macaulay2 and torics easier for String Theorists:
  1. **Kreuzer-Skarke database**
  2. Intersection theory (e.g. using Hirzebruch and Groethendieck - Riemann - Roch)
  3. Includes functions for working on complete intersections of torics varieties
  4. Loads Macaulay2 packages: **Polyhedra**, **NormalToricVarieties**
  5. Interfaces also to **CohomCalg** (Blumenhagen et al) and **Topcom** (Rambau)
  6. Computing maps between cohomology groups

Macaulay2 example

Now let's do a Macaulay2 example: a simplicial toric variety V with 7 rays, of dimension 4, where the Picard rank of the Calabi-Yau hypersurface is $h^{1,1}(X) = 3$.

Methods we know to compute sheaf cohomology on V

Ways we can find these via computer

- (Eisenbud-Mustata-Stillman, 2000): method to compute **any** sheaf cohomology group for any $\mathcal{F}$ on V. Needs: $\mathcal{F}$ given as a graded S-module M, where S is the Cox polynomial ring $S = \mathbb{C}[x_1, \ldots, x_N]$, graded by the Picard group of V. (*Macaulay2*; Martin Bies algorithm and implementation)
- (EMS 2000), (Blumenhagen et al, 2010-2011): method to compute a $\mathbb{C}$-basis of the sheaf cohomology group $H^p(V, \mathcal{O}_V(\sum_{i=1}^{N} a_i \widehat{D}_i))$, as a span of monomials in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$. Beautiful structure! (*Macaulay2*, CohomCalg – more efficient, used by *Macaulay2* when possible)

Main problem: Both methods fail on even reasonably sized examples, e.g. from $h^{11}(X)$ greater than 5 or 10 or so. We need better methods!

Cohomology via stratification

Recall: V is the toric variety corresponding to a given triangulation of Δ^o. Let $D_1, \ldots, D_N$ be the toric divisors on X.

Can we compute $h^\bullet(\mathcal{O}_{D_i})$? **YES!!**

Definition (genus of a face)

Let φ be a face of the polytope Δ^o. φ corresponds to a dual face of Δ. The number of interior lattice points of this dual face is called the (arithmetic) **genus** of φ, denoted by $g(\varphi)$.

Theorem (A. Braun)

Suppose that D_i corresponds to the lattice point $p_i \in \Delta^o$, and that φ is the minimum face of Δ^o containing p_i. Then

1. *If* $\dim \varphi = 0$, *then* $h^\bullet(\mathcal{O}_{D_i}) = [1, 0, g(\varphi)]$,
2. *If* $\dim \varphi = 1$, *then* $h^\bullet(\mathcal{O}_{D_i}) = [1, g(\varphi), 0]$,
3. *If* $\dim \varphi = 2$, *then* $h^\bullet(\mathcal{O}_{D_i}) = [1 + g(\varphi), 0, 0]$.

In the latter case, if $g(\varphi) > 0$, then D_i is the disjoint union of $1 + g(\varphi)$ rigid divisors, no one of which is induced from toric divisors on V.

Geometry of divisors on X

Let $D_1, \ldots, D_N$ be the toric divisors on X (corresponding to lattice points of Δ^o).

Let $\mathcal{T}$ be the simplicial complex consisting of

$$\left\{ \sigma \subset \{1, \ldots, N\} \;\middle|\; \bigcap_{i \in \sigma} D_i \neq \emptyset \right\}$$

$\mathcal{T}$ consists of all simplices in the triangulation of Δ^o which lie on 2-faces (or smaller) of Δ^o.

One way to think of $\mathcal{T}$: Write down all of the 2-faces of the polytope Δ^o, with their induced triangulations. Glue along common edges of these 2-faces.

The Ravioli complex $\mathcal{R}$

Problem: If D_i corresponds to a lattice point interior to a 2-face φ, which has $g(\varphi) > 0$, then D_i is the disjoint union of $1 + g(\varphi)$ *smooth* rigid divisors on X. We want to consider these!

The "ravioli complex" allows us to consider these too!

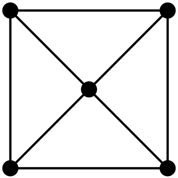
Definition (Ravioli complex $\mathcal{R}$)

The Ravioli complex is obtained by taking, for each 2-face $\varphi \subset \Delta^o$ and its triangulation, $1 + g(\varphi)$ copies of this 2-D simplicial complex. Glue all of these 2-faces along common edges.

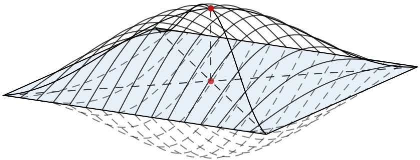
Slight warning: $\mathcal{R}$ isn't necessarily a simplicial complex, it is a *Delta complex*.

Picture of the ravioli complex

Consider a triangulated 2-face of Δ^o with genus 2.



Glue 3 copies of this along their common boundary edges:



Basically the same as $\mathcal{T}$, but puff up all 2-faces of Δ^o with genus > 0.

$Pic(X)$

From now on, let $N = h^{11}(X) + 4$, and let $D_1, \ldots, D_N$ denote all of the connected components of the toric divisors above.

Proposition

The $N = h^{11}(X) + 4$ connected divisors $D_1, \ldots, D_N$ corresponding to the vertices of the Ravioli complex generate $Pic(X)$.

Some combinatorial notions associated to D and Δ^o

Recall: let $D = \sum_{i \in G} D_i$ be a squarefree divisor on X, where $G \subset \{1, \dots, N\}$.

Need the following notions for the main result:

$\mathcal{R}_D$ is the restriction of $\mathcal{R}$ to $\{D_i \mid i \in G\}$.

Some simple combinatorial notions

Let φ be a vertex or edge of Δ^o.

- $\varphi \subset D$ means that every divisor corresponding to a lattice point of φ occurs in D.
- If φ is an edge of Δ^o, $\nu(D, \varphi)$ is the number of connected components of $\mathcal{R}_D|_\varphi$ which do not touch the boundary of φ (φ is an edge or 2-face)

Main theorem

Theorem (Braun, Long, McAllister, -, Sung)

Fix a reflexive 4-polytope Δ, triangulation of Δ^o.

Let $X \subset V$ be the Calabi-Yau 3-fold constructed above.

Let $D_G = \sum_{i \in G} D_i$ be a squarefree divisor on X, and let $\mathcal{R}_D$ be the Ravioli complex above.

Then

- $h^0(\mathcal{O}_D) = h_0(\mathcal{R}_D)$.
- $h^1(\mathcal{O}_D) = h_1(\mathcal{R}_D) + \sum_{\varphi \in \Delta^o(1)} g(\varphi)\nu(D, \varphi)$.
- $h^2(\mathcal{O}_D) = h_2(\mathcal{R}_D) + \sum_{\substack{\varphi \in \Delta^o, \varphi \subset G \\ \dim \varphi \leq 1}} g(\varphi)$

$\dim X = 4$ case mostly understood as well. *Work in progress!*

The reflexive polytope giving the largest rank of $Pic(X) = 491$

In this example, Δ^o is the convex hull of the columns of

$$B = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 2 & -1 \\ -1 & -1 & 6 & -1 & -1 \\ -1 & 83 & -1 & -1 & -1 \end{pmatrix}$$

Δ^o has 679 non-zero lattice points. However 184 of these are interior to 3-faces, and therefore do not intersect X. X has $h^{1,1}(X) = 491$, with 495 toric divisors.

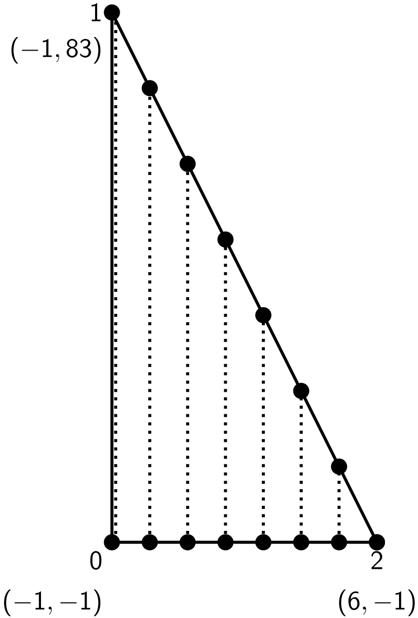
$h^{1,1}(X) = 491$ example

The face structure is very easy here. This is a 4-simplex, with vertices indexed by {0,1,2,3,4}. There are 5 vertices, 10 edges, and 10 triangles (and 5 facets, but we don't care about that here so much).

dim	face	# int pts	genus
0	0	1	0
0	1	1	0
0	2	1	1
0	3	1	3
0	4	1	6
1	0,1	83	0
1	0,2	6	0
1	0,3	2	0
1	0,4	1	0
1	1,2	6	0
1	1,3	2	0
1	1,4	1	0
1	2,3	0	1
1	2,4	0	2
1	3,4	0	6

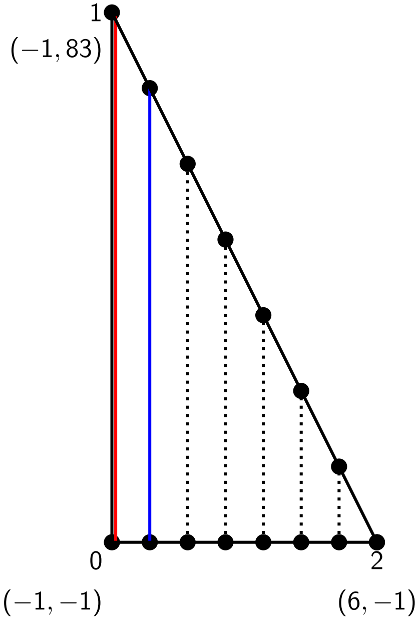
dim	face	# int pts	genus
2	0,1,2	246	0
2	0,1,3	82	0
2	0,1,4	41	0
2	0,2,3	6	0
2	0,2,4	3	0
2	0,3,4	1	0
2	1,2,3	6	0
2	1,2,4	3	0
2	1,3,4	1	0
2	2,3,4	0	1

Rigid divisors on the $h^{11}(X) = 491$ example



- This face has 344 lattice points. All faces except vertex 2 have genus 0.
- Choose a set of points which are connected, and have no cycles. Then this divisor will be rigid.

Rigid divisors on the $h^{11}(X) = 491$ example



- This face has 344 lattice points. All faces except vertex 2 have genus 0.
- Choose a set of points which are connected, and have no cycles. Then this divisor will be rigid.
- take blue divisors and any subset of the 85 red divisors at $x = -1$. Get 2^{85} rigid divisors.

Conclusions

What have we done?

- Introduced *Macaulay2*. Consider trying it, and/or telling me what you would like it to do in this area!
- We considered square-free divisors on $X \subset V$, a CY 3-fold hypersurface in a 4D toric variety.
- For even moderate size $h^{11}(X)$, current algorithms and implementations cannot compute their cohomology.
- We found a formula which can be used to compute these cohomologies (mostly by hand), which works even when $h^{11}(X)$ is large.
- The $h^{11}(X) = 491$ example has 492 smooth rigid divisors, and at least 2^{85} rigid (singular) divisors on it. Probably more like 2^{200}.

Thanks!

Ideal of proof, part 1: Mayer Vietoris sequences

Suppose $D = E + F$. A basic first idea is to use Mayer-Vietoris exact sequences, e.g.

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_E \oplus \mathcal{O}_F \longrightarrow \mathcal{O}_{E\cap F} \longrightarrow 0.$$

and knowledge or induction on the cohomology of the $\mathcal{O}_E$ and $\mathcal{O}_F$. This works for simple cases, but it is hard to control the maps.

So consider the generalized Mayer-Vietoris complex $MV(D_1, \ldots, D_r)$:

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{D_i} \longrightarrow \bigoplus_{i<j} \mathcal{O}_{D_i \cap D_j} \longrightarrow \ldots \longrightarrow \mathcal{O}_{D_1 \cap D_2 \cap \ldots \cap D_r} \longrightarrow 0$$

Proposition

If X is a smooth variety or is a simplicial toric variety, and $D_1, \ldots, D_r$ are effective divisors, such that the intersection of each set of n of these is either empty, or has codimension n in X, then the Mayer-Vietoris sheaf sequence $MV(D_1, \ldots, D_r)$ is an exact sequence of $\mathcal{O}_X$-modules.

Idea of proof, part 2. Hypercohomology

The hypercohomology spectral sequence of $MV(D_1, \ldots, D_r)$ converges to the cohomology $H^n(\mathcal{O}_D)$.

In our case, $\dim X = 3$, and $D = \sum_{i \in G} D_i$ is a graphic divisor, with 2D simplicial complex T_D.

Let $F_0 := \bigoplus_{i \in G} \mathcal{O}_{D_i}$, $F_1 := \bigoplus_{i<j} \mathcal{O}_{D_i \cap D_j}, \ldots$. The E^1 page looks like the following, where the differential is horizontal.

$H^2(F_0)$	0	0
$H^1(F_o)$	$H^1(F_1)$	0
$H^0(F_0)$	$H^0(F_1)$	$H^0(F_2)$

For simplicity, assume that V is **favorable**, i.e. if a 2-face f has an interior lattice point, then $g(f) = 0$. (Not needed, but simpler at the board), and that D lies on a single 2-face, or at least, $H_2(T_D) = 0$.

E^2 page:

$H^2(F_0)$	0	0
$\ker \alpha$	$coker \alpha$	0
$H_0(T_D)$	$H_1(T_D)$	$H_2(T_D) = 0$

Idea of proof, part 3. Argument to show that E^2 differential is zero

- Thus, this spectral sequence has $E^2 = E^\infty$. Now identify combinatorially the terms in this diagram. This gives the theorem.
- If we don't make simplifying assumptions: the spectral sequence still stops here, but the argument is much more delicate.
- What about for $\dim V = 5$? Answer: the spectral sequence doesn't converge until the next page E^3, or perhaps E^4. One can identify all or almost all!) the terms combinatorially. This is still work in progress.