Computational Algebraic Geometry meets String Theory
Rigid divisors and computing sheaf cohomology on Calabi-Yau hypersurfaces of toric 4-folds

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(joint with Andreas Braun, Cody Long, Liam McAllister, Benjamin Sung)
The setting for our talk is a Calabi-Yau hypersurface 3-fold in a 4D simplicial toric variety: \( X \subset V \).

1. Start with a 4D reflexive polytope \( \Delta \subset \mathbb{R}^4 \), and its polar dual \( \Delta^\circ \). Place the single interior point at the origin.

2. Triangulate \( \Delta^\circ \) (using fine, regular, star triangulation).

3. Define a simplicial fan \( \Sigma \): rays are the rays from the origin to (other) lattice points of \( \Delta^\circ \). Cones are those induced by triangulation.

4. \( V := \mathbb{P}_\Sigma \) is a simplicial toric variety, with at most point-like singularities.

5. Let \( F \) be a generic or random sum of monomials corresponding to lattice points of \( \Delta \).

6. Define \( X := (F = 0) \subset V \). \( X \) is our smooth CY3.

Important part to remember: the triangulation of \( \Delta^\circ \).
Rigid divisors and Hodge numbers of divisors on the CY3 $X$

**Definition**

If $D = D_2 \subset X$ is an effective divisor (i.e. a surface, possibly with several components),

- Let $h^i(\mathcal{O}_D) := \dim H^i(D, \mathcal{O}_D)$. The **Hodge vector** of $D$ is
  
  $$h^\bullet(\mathcal{O}_D) := (h^0(\mathcal{O}_D), \ h^1(\mathcal{O}_D), \ h^2(\mathcal{O}_D)).$$

- $D$ is **rigid** if $h^\bullet(\mathcal{O}_D) = (1, 0, 0)$ (not necessarily smooth or irreducible!)

**Computational problems of interest**

- **Our main objective!** Given an effective divisor $D \subset X$, find $h^\bullet(\mathcal{O}_D)$.
- Given $X$, are there finitely many or infinitely many rigid divisors on $X$? (either case is interesting).
- Given $X$, computationally find all, or many, rigid divisors on $X$. 
One goal for this talk

- Let $\Delta^o$, $X \subset V = \mathbb{P}_\Sigma$ be as just constructed.
- Let the $\widehat{D}_i$ be the torus invariant prime divisors on $V$. These are in 1-1 correspondence with the (non-zero) lattice points of $\Delta^o$.
- Let $D_i := \widehat{D}_i \cap X$. $D_i$ is nonempty exactly when the lattice point lies on a 2-face of $\Delta^o$, that is, is not internal to a facet of $\Delta^o$.
- Let $D_1, \ldots, D_N$ be the collection of all of these divisors on $X$.

**Key problem**

Let $D$ be a **square-free divisor**, i.e. $D = \sum_{i \in G} D_i$, where $G \subset \{1, \ldots, N\}$.

**Goal:** Compute the Hodge numbers

\[ h^\bullet(\mathcal{O}_D) := (h^0(\mathcal{O}_D), \ h^1(\mathcal{O}_D), \ h^2(\mathcal{O}_D)) \, . \]

Actually: will consider a somewhat larger class of divisors, which includes these.
Computing with these polyhedra and Calabi-Yau varieties using Macaulay2

- **Macaulay2.** An open source computer algebra system by Dan Grayson and myself, for investigations in algebraic geometry and related fields.
  - 1 Dan: one of the 7 original authors of Mathematica
  - 2 A community project: over 100 – 150 user written packages in Macaulay2

- **StringTorics.** A new **Macaulay2** package, almost complete, designed to make the use of Macaulay2 and torics easier for String Theorists:
  - 1 Kreuzer-Skarke database
  - 2 Intersection theory (e.g. using Hirzebruch and Groethendieck - Riemann - Roch)
  - 3 Includes functions for working on complete intersections of torics varieties
  - 4 Loads Macaulay2 packages: **Polyhedra**, **NormalToricVarieties**
  - 5 Interfaces also to **CohomCalg** (Blumenhagen et al) and **Topcom** (Rambau)
  - 6 Computing maps between cohomology groups
Now let’s do a Macaulay2 example: a simplicial toric variety $V$ with 7 rays, of dimension 4, where the Picard rank of the Calabi-Yau hypersurface is $h^{1,1}(X) = 3$. 
Methods we know to compute sheaf cohomology on $V$

Ways we can find these via computer

- (Eisenbud-Mustata-Stillman, 2000): method to compute any sheaf cohomology group for any $\mathcal{F}$ on $V$. Needs: $\mathcal{F}$ given as a graded $S$-module $M$, where $S$ is the Cox polynomial ring $S = \mathbb{C}[x_1, \ldots, x_N]$, graded by the Picard group of $V$. (*Macaulay2; Martin Bies algorithm and implementation*)

- (EMS 2000), (Blumenhagen et al, 2010-2011): method to compute a $\mathbb{C}$-basis of the sheaf cohomology group $H^p(V, \mathcal{O}_V(\sum_{i=1}^{N} a_i \hat{D}_i))$, as a span of monomials in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$. Beautiful structure! (*Macaulay2, CohomCalg – more efficient, used by Macaulay2 when possible*)

Main problem: Both methods fail on even reasonably sized examples, e.g. from $h^{11}(X)$ greater than 5 or 10 or so. We need better methods!
Cohomology via stratification

Recall: $V$ is the toric variety corresponding to a given triangulation of $\Delta^\circ$. Let $D_1, \ldots, D_N$ be the toric divisors on $X$.

Can we compute $h^\bullet(\mathcal{O}_{D_i})$? **YES!!**

**Definition (genus of a face)**

Let $\varphi$ be a face of the polytope $\Delta^\circ$. $\varphi$ corresponds to a dual face of $\Delta$. The number of interior lattice points of this dual face is called the (arithmetic) **genus** of $\varphi$, denoted by $g(\varphi)$.

**Theorem (A. Braun)**

*Suppose that $D_i$ corresponds to the lattice point $p_i \in \Delta^\circ$, and that $\varphi$ is the minimum face of $\Delta^\circ$ containing $p_i$. Then*

1. If $\dim \varphi = 0$, then $h^\bullet(\mathcal{O}_{D_i}) = [1, 0, g(\varphi)]$,
2. If $\dim \varphi = 1$, then $h^\bullet(\mathcal{O}_{D_i}) = [1, g(\varphi), 0]$,
3. If $\dim \varphi = 2$, then $h^\bullet(\mathcal{O}_{D_i}) = [1 + g(\varphi), 0, 0]$.

In the latter case, if $g(\varphi) > 0$, then $D_i$ is the disjoint union of $1 + g(\varphi)$ rigid divisors, no one of which is induced from toric divisors on $V$. 
Let $D_1, \ldots, D_N$ be the toric divisors on $X$ (corresponding to lattice points of $\Delta^o$).

Let $\mathcal{T}$ be the simplicial complex consisting of

\[
\left\{ \sigma \subset \{1, \ldots, N\} \mid \bigcap_{i \in \sigma} D_i \neq \emptyset \right\}
\]

$\mathcal{T}$ consists of all simplices in the triangulation of $\Delta^o$ which lie on 2-faces (or smaller) of $\Delta^o$.

One way to think of $\mathcal{T}$: Write down all of the 2-faces of the polytope $\Delta^o$, with their induced triangulations. Glue along common edges of these 2-faces.
Problem: If $D_i$ corresponds to a lattice point interior to a 2-face $\varphi$, which has $g(\varphi) > 0$, then $D_i$ is the disjoint union of $1 + g(\varphi)$ smooth rigid divisors on $X$. We want to consider these!

The “ravioli complex” allows us to consider these too!

**Definition (Ravioli complex $\mathcal{R}$)**

The Ravioli complex is obtained by taking, for each 2-face $\varphi \subset \Delta^o$ and its triangulation, $1 + g(\varphi)$ copies of this 2-D simplicial complex. Glue all of these 2-faces along common edges.

Slight warning: $\mathcal{R}$ isn’t necessarily a simplicial complex, it is a *Delta complex*. 
Consider a triangulated 2-face of $\Delta^o$ with genus 2.

Glue 3 copies of this along their common boundary edges:

Basically the same as $\mathcal{T}$, but puff up all 2-faces of $\Delta^o$ with genus $> 0$. 
From now on, let $N = h^{11}(X) + 4$, and let $D_1, \ldots, D_N$ denote all of the connected components of the toric divisors above.

**Proposition**

The $N = h^{11}(X) + 4$ connected divisors $D_1, \ldots, D_N$ corresponding to the vertices of the Ravioli complex generate Pic($X$).
Recall: let $D = \sum_{i \in G} D_i$ be a squarefree divisor on $X$, where $G \subset \{1, \ldots, N\}$.

Need the following notions for the main result:

$\mathcal{R}_D$ is the restriction of $\mathcal{R}$ to $\{D_i \mid i \in G\}$.

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**Some simple combinatorial notions**

Let $\varphi$ be a vertex or edge of $\Delta^o$.

- $\varphi \subset D$ means that every divisor corresponding to a lattice point of $\varphi$ occurs in $D$.

- If $\varphi$ is an edge of $\Delta^o$, $\nu(D, \varphi)$ is the number of connected components of $\mathcal{R}_D|_{\varphi}$ which do not touch the boundary of $\varphi$ ($\varphi$ is an edge or 2-face)
Theorem (Braun, Long, McAllister, -, Sung)

Fix a reflexive 4-polytope $\Delta$, triangulation of $\Delta^\circ$.

Let $X \subset V$ be the Calabi-Yau 3-fold constructed above.

Let $D_G = \sum_{i \in G} D_i$ be a squarefree divisor on $X$, and let $\mathcal{R}_D$ be the Ravioli complex above.

Then

1. $h^0(\mathcal{O}_D) = h_0(\mathcal{R}_D)$.
2. $h^1(\mathcal{O}_D) = h_1(\mathcal{R}_D) + \sum_{\varphi \in \Delta^\circ(1)} g(\varphi) \nu(D, \varphi)$.
3. $h^2(\mathcal{O}_D) = h_2(\mathcal{R}_D) + \sum_{\varphi \in \Delta^\circ, \varphi \subseteq G} g(\varphi) \sum_{\dim \varphi \leq 1}$

$\dim X = 4$ case mostly understood as well. Work in progress!
The reflexive polytope giving the largest rank of $\text{Pic}(X) = 491$

In this example, $\Delta^o$ is the convex hull of the columns of $B = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 2 & -1 \\ -1 & -1 & 6 & -1 & -1 \\ -1 & 83 & -1 & -1 & -1 \end{pmatrix}$

$\Delta^o$ has 679 non-zero lattice points. However 184 of these are interior to 3-faces, and therefore do not intersect $X$. $X$ has $h^{1,1}(X) = 491$, with 495 toric divisors.
The face structure is very easy here. This is a 4-simplex, with vertices indexed by \{0,1,2,3,4\}. There are 5 vertices, 10 edges, and 10 triangles (and 5 facets, but we don’t care about that here so much).

\[
h^{1,1}(X) = 491\]
Rigid divisors on the $h^{11}(X) = 491$ example

This face has 344 lattice points. All faces except vertex 2 have genus 0.

Choose a set of points which are connected, and have no cycles. Then this divisor will be rigid.
Rigid divisors on the $h^{11}(X) = 491$ example

This face has 344 lattice points. All faces except vertex 2 have genus 0.

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Take blue divisors and any subset of the 85 red divisors at $x = -1$. Get $2^{85}$ rigid divisors.
What have we done?

- Introduced *Macaulay2*. Consider trying it, and/or telling me what you would like it to do in this area!
- We considered square-free divisors on $X \subset V$, a CY 3-fold hypersurface in a 4D toric variety.
- For even moderate size $h^{11}(X)$, current algorithms and implementations cannot compute their cohomology.
- We found a formula which can be used to compute these cohomologies (mostly by hand), which works even when $h^{11}(X)$ is large.
- The $h^{11}(X) = 491$ example has 492 smooth rigid divisors, and at least $2^{85}$ rigid (singular) divisors on it. Probably more like $2^{200}$. 
Thanks!
Suppose $D = E + F$. A basic first idea is to use Mayer-Vietoris exact sequences, e.g.

$$0 \rightarrow \mathcal{O}_D \rightarrow \mathcal{O}_E \oplus \mathcal{O}_F \rightarrow \mathcal{O}_{E \cap F} \rightarrow 0.$$ 

and knowledge or induction on the cohomology of the $\mathcal{O}_E$ and $\mathcal{O}_F$. This works for simple cases, but it is hard to control the maps.

So consider the generalized Mayer-Vietoris complex $MV(D_1, \ldots, D_r)$:

$$0 \rightarrow \mathcal{O}_D \rightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{D_i} \rightarrow \bigoplus_{i<j} \mathcal{O}_{D_i \cap D_j} \rightarrow \cdots \rightarrow \mathcal{O}_{D_1 \cap D_2 \cap \ldots \cap D_r} \rightarrow 0$$

Proposition

If $X$ is a smooth variety or is a simplicial toric variety, and $D_1, \ldots, D_r$ are effective divisors, such that the intersection of each set of $n$ of these is either empty, or has codimension $n$ in $X$, then the Mayer-Vietoris sheaf sequence $MV(D_1, \ldots, D_r)$ is an exact sequence of $\mathcal{O}_X$-modules.
Idea of proof, part 2. Hypercohomology

The hypercohomology spectral sequence of $MV(D_1, \ldots, D_r)$ converges to the cohomology $H^n(\mathcal{O}_D)$.
In our case, $\dim X = 3$, and $D = \sum_{i \in G} D_i$ is a graphic divisor, with 2D simplicial complex $T_D$.
Let $F_0 := \bigoplus_{i \in G} \mathcal{O}_{D_i}$, $F_1 := \bigoplus_{i<j} \mathcal{O}_{D_i \cap D_j}, \ldots$. The $E^1$ page looks like the following, where the differential is horizontal.

$$
\begin{array}{ccc}
H^2(F_0) & 0 & 0 \\
H^1(F_0) & H^1(F_1) & 0 \\
H^0(F_0) & H^0(F_1) & H^0(F_2)
\end{array}
$$

For simplicity, assume that $V$ is favorable, i.e. if a 2-face $f$ has an interior lattice point, then $g(f) = 0$. (Not needed, but simpler at the board), and that $D$ lies on a single 2-face, or at least, $H_2(T_D) = 0$.

$E^2$ page:

$$
\begin{array}{ccc}
H^2(F_0) & 0 & 0 \\
\ker \alpha & \coker \alpha & 0 \\
H_0(T_D) & H_1(T_D) & H_2(T_D) = 0
\end{array}
$$
Idea of proof, part 3. Argument to show that $E^2$ differential is zero

Thus, this spectral sequence has $E^2 = E^\infty$. Now identify combinatorially the terms in this diagram. This gives the theorem.

If we don't make simplifying assumptions: the spectral sequence still stops here, but the argument is much more delicate.

What about for dim $V = 5$? Answer: the spectral sequence doesn’t converge until the next page $E^3$, or perhaps $E^4$. One can identify all or almost all!) the terms combinatorially. This is still work in progress.