Describing algebraic varieties to a computer

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Outline

- Degree of a curve
- Component membership test
- Monodromy and trace test
- Witness sets
- References:
    - Multiprojective witness sets and a trace test
        - Jonathan Hauenstein and [-]
    - Trace test
        - Anton Leykin, [-], Frank Sottile
    - The maximum likelihood degree of mixtures of independence models
        - [-] and Botong Wang (Counting critical points)

Testing membership

Intersect a curve $\mathscr{V}$ with a hyperplane $\mathscr{L}$.

- Consider a curve $\mathscr{V} \subset \mathbb{C}^n$ (may be reducible). [Draw]
- For a general hyperplane $\mathscr{L}$ of $\mathbb{C}^n$, [Draw]
  the degree of $\mathscr{V}$ is the cardinality of $\mathscr{L} \cap \mathscr{V}$.
- Question (Membership): Given a point $B \in \mathbb{C}^n$, is B in the $\mathscr{V}$?
- If we know that equations $F = 0$ define $\mathscr{V}$, then check $F(B) = 0$.
- Membership test: Let $\mathscr{M}$ denote a general hyperplane through B.
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  Problem: How to sort the points $\mathscr{V} \cap \mathscr{L}$ by irreducible component?

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Monodromy

Deforming $\mathscr{L}$ to itself is a monodromy used to sort the points in $\mathscr{V} \cap \mathscr{L}$.

- Repeatedly deform $\mathscr{L}$ to itself to find points in $\mathscr{L} \cap \mathscr{V}$ that are on the same component.
- In example C's go to C's and D's go to D's.
  - Illustrate on circle (Short move then rotation).
- Problem: No stopping criteria to say that we are done sorting.
  - (Reminder: we are working over the complex numbers.)
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Trace

The trace of a set of points S is the coordinate-wise average.

- Let $\mathscr{L}_t$ denote a family of hyperplanes defined by $H + t = 0$.
- Let $\mathscr{X}$ be an irreducible component of $\mathscr{V} \subset \mathbb{C}^n$.
- Suppose $S \subseteq \mathscr{X} \cap \mathscr{L}_t$.
- [Trace test] The trace of S is affine linear in t if and only if $S = \mathscr{X} \cap \mathscr{L}_t$.
  - Numerical irreducible decomposition using projections from points on the components by A.J. Sommese, J. Verschelde, and C.W. Wampler.
- [Draw to illustrate affine linear in t]
- Not limited to curves. Take $\mathscr{L}$ to be defined by linear equations $H_1, H_2, \ldots, H_{\dim \mathscr{V}}$.

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Intersect a curve $\mathscr{V}$ with a hyperplane $\mathscr{L}$ with the intersection sorted by component.

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  - Deform $\mathscr{L} \to \mathscr{M}$ to determine $\mathscr{M} \cap \mathscr{X}$ from $\mathscr{L} \cap \mathscr{X}$.
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Witness sets

We use witness sets to describe algebraic varieties to a computer.

- A witness set for a d dimensional algebraic variety $\mathscr{X}$ in $\mathbb{C}^n$ is a triple of information:
  1. Equations $F = 0$ that define a variety containing $\mathscr{X}$ as an irreducible component.
  2. Linear equations $H_1 = 0, \ldots, H_d = 0$ defining a general codimension d linear space $\mathscr{L}$.
  3. Witness points: Numerical approximations to the points in $\mathscr{X} \cap \mathscr{L}$.
- Advantages:
  - Sample points.
  - Test membership.
  - Refine approximations.
- Computational Savings (two line summary):
  - Monodromy homotopy for partial information
  - Decomposition for focus.

Systems with groups of indeterminants

Many systems define varieties naturally in $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \cdots \times \mathbb{C}^{n_k}$.

- Our results: develop a trace test for these varieties.
- Euclidean distance degree $\mathbb{C}^n_x \times \mathbb{C}^n_u$.
- Method of moments $\mathbb{C}^n_{\mu,\sigma} \times \mathbb{C}^r_a \times \mathbb{C}^n_m$.
  - Found $248,400$ solutions with monodromy (special Galois group).
- Alt's problem in kinematics $\mathbb{C}^1 \times \mathbb{C}^{12}$.
- Tensor completion: 21,288,960 solutions.
- Likelihood equations: $\mathbb{C}^n_p \times \mathbb{C}^n_u$.
- Optimization: primal variables and Lagrange multipliers $\mathbb{C}^n_x \times \mathbb{C}^m_\lambda$.

How do we find $\mathscr{L} \cap \mathscr{V}$?

Regeneration is an equation by equation technique for solving systems of equations.

- Consider

$$F := x_1^2 - x_2$$
$$H := 2x_1 + 3x_2 + 7$$

- $F = 0$ defines a curve $\mathscr{V}$.
- H defines $\mathscr{L}$.
- We begin with linear algebra and regenerate to nonlinear systems.
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Our results

- "Multiprojective witness sets and a trace test"
    - We introduce witness sets for multiprojective varieties
    - A trace test for multiprojective varieties.
    - A membership test for multiprojective varieties
- "Trace test" Twelve pages!
    - We have a dimension reduction for $\mathbb{C}^n \times \mathbb{C}^m$ to $\mathbb{C}^1 \times \mathbb{C}^1$
- Contact: "Jose Israel Rodriguez"
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    - http://home.uchicago.edu/~joisro/