Fourier Mukai Transformation of Vector Bundles over Blown Up spaces

String Phenomenology
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• Motivations

• Fourier Mukai transform describe the moduli space of stable vector bundle as a fibration

\[ \mathcal{Y} \rightarrow \mathcal{M} \]

\[ \mathcal{M} : \text{Moduli space of some n-sheeted cover of the base} \]

\[ \mathcal{Y} : \text{Moduli space of a line bundle over that cover} \]

Moduli space of Het \hspace{1cm} \text{FM} \hspace{1cm} \text{Moduli space of F-theory}
Motivations

• Fourier Mukai transform describe the moduli space of stable vector bundle as a fibration

$$\mathcal{Y} \to \mathcal{M}$$

- \(\mathcal{M}\) : Moduli space of some n-sheeted cover of the base
- \(\mathcal{Y}\) : Moduli space of a line bundle over that cover

• It is possible compute the cover, but the line bundle over that is not well understood. [Bershadsky et. al, hep-th/9712023]

• The obstacle is, limited information about the divisors inside the n-sheeted cover.

• Previous works can be somewhat extended by studying the contribution of the induced divisors on the spectral cover from exceptional divisors in the heterotic CY.

FM transform over Blown up Varieties
• **Het/F Duality Review**

Heterotic $E_8 \times E_8$ to $F$-Theory

$X_{n+1}$

$\begin{bmatrix}
T^2 \\
B_n
\end{bmatrix}$

$X_{n+2}$

$\begin{bmatrix}
\text{Elliptically Fibered } K3 \\
B_n
\end{bmatrix}$

• **Volume of $B_n$ should be large relative to the torus,** (Adiabatic Approximation)

Vafa 96, Evidence for F-theory

Vafa, Witten hep-th/9507050

• $N = 1$ SUSY in effective theory requires,
  • $X_{n+1}$ and $X_{n+2}$ are both CY
  • $V$ is Stable

• $C_1(V) = 0$ for subgroups of $E_8$
Spectral Cover Construction

Theorem (HN): $V_{E_p}$ is semistable for a generic elliptic curve.

Theorem (Atiyah 57): a semistable vector bundle over a smooth elliptic curve decomposes as

$$V_E = \bigoplus_{i=1}^{N} O(p_i - \sigma)$$

$$C_1(V) = 0 \Rightarrow \sum_{i} p_i = 0$$
• Spectral Cover Construction

FMW hep-th/9701162
FMW alg-geom/9707004

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• We need another piece of data called **spectral sheaf** or **spectral line bundle**

$$ \pi_1 \quad X \times_{B_n} \hat{X} \quad \pi_2 $$

$$ FM^1(V) = R^1\pi_2^*(\pi_1^*V \otimes \tilde{\mathcal{P}}) $$

$$ FM^1(V)|_S = \mathcal{L} $$

$$ \hat{F}M^0(\mathcal{L}) = \pi_1^*(\pi_2^*\mathcal{L} \otimes \mathcal{P}) = V $$
• Chern Classes

• Chern classes can be computed by Grothendieck-Riemann-Roch

\[ \pi_S : S \to B \]

\[ \pi_S^*(\exp(C_1(\mathcal{L}) + C_1(\mathcal{P}_B))Td(S \times_B X)) = Ch(V)Td(X) \]

• If one assume \( C_1(\mathcal{L}) \) only depends on \( \sigma \cap S \) and \( \pi_S^* D_\sigma \), the following results can be found: \([S] = n \sigma + \eta\)

\[ C_1(\mathcal{L}) = \frac{1}{2}(\pi_S^*C_1(\mathcal{B}) - C_1(S)) + \lambda(n\sigma - (\eta - nC_1(\mathcal{B})) \]

\[ C_2(V) = \sigma \eta + \omega \]

\[ C_3(V) = 2\lambda \eta(\eta - nC_1(\mathcal{B})) \]
Matter Content

By Lerray,

\[ H^1(X, V) = H^0(B, R^1\pi_* V) \]

\[ R^1\pi_*(\pi_2^*(\pi_1^*\mathcal{L} \otimes \mathcal{P})) = \pi_S^*(R^1\pi_1^*(\pi_1^*\mathcal{L} \otimes \mathcal{P})) = \pi_S^*(\mathcal{L} \otimes R^1\pi_1^*\mathcal{P}) \]

\[ R^1\pi_1^*(i^*\mathcal{P}) = i^*R^1\pi_X^*\mathcal{P} = i^*R^1\pi_X^*\mathcal{O}_{X \times X} = i^*\sigma^*K_B \]

\[ H^0(B, R^1\pi_* V) = H^0(\sigma \cap S, \mathcal{L} \otimes K_B) \]
• SU(2) singularity with Holomorphic section

• Consider a generic double cover $\Rightarrow$ Double point singularities of the branch curve induce double point singularity on the spectral cover.
• **SU(2) singularity with Holomorphic section**

- Consider a generic double cover ➔ Double point singularities of the branch curve induce double point singularity on the spectral cover.
- If these singularities coincide with the singularities of the $CY_3$, the blow ups induce different (-2) curves on the double cover.

\[
\begin{array}{cccccc|c}
3 & 2 & 1 & 0 & 0 & 0 & 6 \\
9 & 6 & 0 & 1 & 1 & 1 & 18 \\
\end{array}
\rightarrow
\begin{array}{cccccc|c}
3 & 2 & 1 & 0 & 0 & 0 & 0 \\
8 & 5 & 0 & 1 & 1 & 1 & 16 \\
9 & 6 & 0 & 0 & 1 & 1 & 18 \\
\end{array}
\]
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• After finding \( \widetilde{X} \times_B \widetilde{X} \) (by blowing up \( X \times_B X \)), we can use the GRR theorem to compute the topology of the corresponding SU(2) bundle.

• The result for the special examples \([s] = 2\sigma + 7D\) is,

\[
C_1(\mathcal{L}) = \sigma + 5D + \lambda(2\sigma - D) + \sum_{i=1}^{7} \beta_i E_i
\]

\[
C_2(V) = 7\sigma D + (7\lambda^2 + \sum_{i=1}^{7} \beta_i^2 - 22)D^2 + \left(\sum_{i=1}^{7} \beta_i^2 - 2\sum_{i=1}^{7} \beta_i + 6\right)E \cdot D
\]
SU(2) singularity with Holomorphic section

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- After finding $\tilde{X} \times_B \tilde{X}$ (by blowing up $X \times_B X$), we can use the GRR theorem to compute the topology of the corresponding SU(2) bundle.
- The result for the special examples $[s] = 2\sigma + 7D$ is,

$$C_1(L) = \sigma + 5D + \lambda(2\sigma - D) + \sum_{i=1}^{7} \beta_i E_i$$

$$C_2(V) = 7\sigma D + (7\lambda^2 + \sum_{i=1}^{7} \beta_i^2 - 22)D^2 + (\sum_{i=1}^{7} \beta_i^2 - 2 \sum_{i=1}^{7} \beta_i + 6)E \cdot D$$

- Holomorphic zero section doesn’t intersect with singularities, therefore $\sigma \cdot E_i = 0$ and there is no new contribution in the matter content of EFT.
• **SU(2) singularity with Rational section**

• Consider a case that zero section wraps around a (-2)-curve after blow up.

\[ \sigma \cdot E \neq 0 \]

\[
\begin{array}{cccc|c}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 \\
1 & 1 & 1 & 0 & 2 & 3 & 0 & 8 \\
\hline
\end{array}
\]

\[
\begin{array}{cccc|c}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 \\
1 & 1 & 1 & 0 & 0 & 2 & 3 & 0 & 8 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 5 \\
\hline
\end{array}
\]

• In this case we have two sections \( \sigma_1 = (1,0,0) \) and \( \sigma_2 = (-1,1,2) \), the first one is isomorphic to the base. But after blow up it wraps around a (-2)-curve.
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\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c
\end{array}
\]

- In this case we have two sections \( \sigma_1 = (1,0,0) \) and \( \sigma_2 = (-1,1,2) \), the first one is isomorphic to the base. But after blow up it wraps around a (-2)-curve.
- By repeating the same arguments we get the following result for \([S] = n \sigma_1 + k D\)

\[
C_1(\mathcal{L}) = \frac{1}{2}(C_1(B_2) - C_1(S)) + \lambda_1(n \sigma_1 - (k - 3n)D + E_\sigma) + \lambda_2(n \sigma_2 - (5n + k)D + (4k(n - 1) + n)E_\sigma) + \sum_{i=1}^{k} \beta_i E_i + E_\sigma
\]

- \( \chi(X,V) \), changes by \( 2n (\lambda_1 + \lambda_2 (4k - n) + 1) \).

- \( C_2(V) \) and \( C_3(V) \) are are complicated functions which depend on the coefficients of \( E_i \) and \( E_\sigma \).
• Conclusion

- If the exceptional divisor induce new (-2)-curves, then topology of bundle changes.

- Matter content changes only if zero section wraps around some (-2)-curve induced by the exceptional divisor.

- The next questions will be
  1. How it’s possible to see the change in matter content from the F-theory dual.
  2. The contribution of new discrete set of data $\left( \sum_i \beta_i E_i \right)$ in $G_4$ flux should be studied.
• Conclusion

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  1. How it’s possible to see the change in matter content from the F-theory dual.

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Thank you