**Extended Moduli Spaces**

and a corresponding **Moduli Space Size Conjecture**

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based on work with Philipp Henkenjohann and Lukas Witkowski

**Outline**

- Recall the Weak Gravity Conjecture for axions: $f < M_P$.

- We try to circumvent this extending the moduli space with fluxes (‘winding trajectories’).

- If we do not address inflation, SUSY-breaking, moduli-stabilization, this can be done very explicitly.

- Nevertheless, a ‘Moduli Space Size Conjecture’ appears to hold.
Introduction

• The Weak Gravity Conjecture,

\[ m < gM_P \quad \text{or} \quad \Lambda < gM_P , \]

has recently been revisited by many authors:

Cheung/Remmen; Rudelius; de la Fuente/Saraswat/Sundrum . . . ’14
Ibanez/Montero/Uranga/Valenzuela; Brown/Cottrell/Shiu/Soler;
Bachlechner/Long/McAllister; AH/Mangat/Rompineve/Witkowski;
Junghans; Heidenreich/Reece/Rudelius; Kooner/Parameswaran/Zavala;
Harlow; AH/Rompineve/Westphal; . . . ’15
Conlon/Krippendorf; Ooguri/Vafa; Freivogel/Kleban; Banks;
Danielsson/Dibitetto; . . . . . . ’16
Introduction (continued)

- For recent work concerning the derivation of the WGC in various contexts see e.g.
  
  Cottrell/Shiu/Soler '16
  Fisher/Mogni '17
  Soler/Hebecker '17
  Hod '17
Motivation (continued)

- A particularly timely aspect of it is the axionic case, 
  \[ g \equiv \frac{1}{f}, \]
  relevant for natural inflation.

- Another important motivation: 
  Learning general lessons about quantum gravity.

- Expect relations to Ooguri-Vafa swampland conjecture 
  ['Going long distances in moduli space lowers the cutoff 
  exponentially.‘]
  Ooguri/Vafa, ’05, ’06 
  (see also Klaewer/Palti, ’16)
Let us first recall the **Generalized WGC:**

**General Action:**

\[ S \sim \int d^4 g^2 F_{p+1}^2 + \int A_p + T \int_p (\ast 1) \]

**WGC:**

\[ g > \frac{T}{M_{P}^{d/2-1}} \sim \frac{\Lambda^p}{M_{P}^{d/2-1}} \]

- Specifically for an axion in \( d = 4 \) this implies

\[ \frac{1}{f} > \frac{S_{\text{inst.}}}{M_{P}} \text{ or even } f < M_{P} \]

- This case is very special since the cutoff \( \Lambda \) drops out. But this is too quick – we will see at the end that \( \Lambda \) makes a comeback.
It is known that:

- $f < M_P$ is consistent with all simple stringy examples.  
  \[ \text{Banks et al. '03} \]

- It is consistent in spirit with the swampland conjecture ('no large distances in moduli space').  
  \[ \text{see especially Klaewer/Palti '16} \]

- It is challenged by Monodromy.  
  \[ \text{McAllister/Silverstein/Westphal} \]

- It is also challenged by KNP.  
  \[ \text{Kim/Nilles/Peloso '04} \]

- Here, we want to use the ‘Winding inflation’ realization of the last idea to see whether we can beat the WGC for axions.  
  \[ \text{AH/Mangat/Rompineve/Witkowski} \]
• Even in a small field space a long trajectory can be realized if the potential is appropriate.

Kim/Nilles/Peloso '04 (Berg/Pajer/Sjors '09; Ben-Dayan/Pedro/Westphal '14)

The possibly simplest way to achieve this is via gauging à la Dvali (cf. also KS/KLS), as in ‘Winding Inflation’.

AH/Mangat/Rompineve/Witkowski '14

\[ |F_0|^2 \rightarrow |F_0 + \varphi_x + N\varphi_y|^2 \]

• This is can be realized very explicitly in the flux landscape, with \( N \) being the flux number.
An Aside:

- Recently, the same gauging idea of Dvali has been discussed as a way to evade the WGC for 1-forms. Saraswat '16

- Our personal feeling is that
  (a) This is very interesting to explore further.
  (b) In the end, it won’t work since the UV theory will not permit \( N \gg 1 \) together with \( \Lambda \sim M_P \), as required.

- The technical reason might be as follows:

  \[
  N \gg 1 \quad \Rightarrow \quad \text{Ratio of certain radii is large (e.g. } R_A/R_B \gg 1) \\
  \Rightarrow \quad \Lambda \ll M_P.
  \]

  (This logic is not applicable in the axionic case since \( \Lambda \) does not enter. We may have an interesting answer to this....)
Our example:

• Type IIB on $T^6/\mathbb{Z}_2$ with 64 O3 planes.
• Using standard technology, we can generate

$$W = (M\tau_1 - N\tau_2)(\tau - \tau_3)$$

Kachru/Schulz/Trivedi '02
Gomis/Marchesano/Mateos '05
...

(The explicit $F_3/H_3$ will appear in a moment.)

• $D_{\tau_i} W = 0$ ensure $W = 0$ together with

$$M\tau_1 = N\tau_2 \quad \text{and} \quad \tau = \tau_3.$$

• If, for example, $M = 1, N \gg 1$, this gives exactly our previous winding picture with

$$\varphi_x \equiv \text{Re}\tau_1 \quad \text{and} \quad \varphi_y \equiv \text{Re}\tau_2.$$
Comments:

- Many authors have considered monodromy & backreaction.
- Back-reaction induced, logarithmic limits on field-space distances have been in particular been suggested by Klaewer/Palti '16.
- What we do here is very different:
  
  (1) No real monodromy – just an extended periodic field space.
  (2) No backreaction – our field space is ‘SUSY Minkowski’.
  
  Still, a logarithm will emerge...

- Recent work related in spirit includes...
  
  Bielleman/Ibanez/Valenzuela '15
  Conlon/Krippendorf '16
• We will ignore $\tau, \tau_3$ and all Kahler moduli.  
  (We do not care about pheno - only about the WGC.)

• On the 4-dimensional $\tau_1/\tau_2$ moduli space, we have the constraint $\tau_1 = N\tau_2$.

• Parameterize the remaining 2-dimensional space using just $\tau_1$:

  $$\mathcal{L} \supset \frac{(\partial \tau_1)^2}{|\tau_1 - \bar{\tau}_1|^2} + \frac{(\partial \tau_2)^2}{|\tau_2 - \bar{\tau}_2|^2} \sim \frac{(\partial \phi)^2}{\text{Im}\tau_1^2}$$

  with $\phi \equiv \text{Re}\tau_1 \in (-N/2, N/2)$.

• With the tadpole constraint $MN \leq 16$, this allows us $N = 16$ and hence, with $\text{Im}\tau_1 \simeq 1$ we get $f_{\text{eff}}/M_P \simeq 16$.

  (Much more should be doable on CYs in the large-complex-structure limit.)
Before claiming victory, we should revisit the other moduli.

Dismissing $\tau, \tau_3$ and Kahler moduli may be OK – their spaces factorize. But $\text{Im}\tau_1$ is really part of our game...

Most naively, $\tau_1$ describes $T^2$ and lives in the fundamental domain of $SL(2, \mathbb{Z})$.

Of course, we already know that the horizontal periodicity must somehow be enlarged $N$ times.
To make this explicit, let us spell out the flux:

\[ F_3 = (-M \, dx_1 \wedge dy_2 + N \, dy_1 \wedge dx_2) \wedge dx_3 = +A \wedge dx_3 \]
\[ H_3 = (+M \, dx_1 \wedge dy_2 - N \, dy_1 \wedge dx_2) \wedge dy_3 = -A \wedge dy_3. \]

The 2-form \( A \) lives only on the first two tori:

\[ A = A_{ij} \, d\xi^i_1 \wedge d\xi^j_2 \quad \text{with} \quad \xi^i_{1,2} = \begin{pmatrix} y_{1,2} \\ x_{1,2} \end{pmatrix}. \]

The essential flux information is in the matrix

\[ A_{ij} = \begin{pmatrix} 0 & N \\ -M & 0 \end{pmatrix}. \]
Under $R_1 \in SL(2, \mathbb{Z})$, the $T_1^2$ and the flux transform as

$$\tau_1 \rightarrow \tau_1' = R_1(\tau_1) = \frac{a\tau_1 + b}{c\tau_1 + d},$$

and

$$A \rightarrow A' = R_1A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & N \\ -M & 0 \end{pmatrix}.$$

To map this back to the original configuration, we need $R_2 \in SL(2, \mathbb{Z})$ of $T_2^2$:

$$A' = R_1AR_2^T = A$$

But this is only possible if $b = 0 \text{ (mod } N)$ and $c = 0 \text{ (mod } M)$.

In mathematical terms: $R_1$ must be in one of the Congruence Subgroups of $SL(2, \mathbb{Z})$. 
- These subgroups have a larger fundamental domain, corresponding to the Extended Moduli Spaces of fluxed tori.

- As a simple example, consider $M = 1$ and $N = 5$, leading to the congruence subgroup $\Gamma^0(5)$ with fundamental domain:

- The horizontal extension at $\text{Im} \tau_1 \gg 1$ was of course expected, but the structure near the real axis can be complicated...
• To appreciate this, consider e.g. part of the domain of $\Gamma^0(7)$, with the appropriate identifications indicated:

Helena A. Verrill, 2001
see also her code ‘fundomain’

• A sketch of the actual full geometry of such extended moduli spaces might look as follows:
• Let us finally look at a case where the ‘upper’ throat \((cusp)\) is extended even more, \(N = 12\).

![Diagram of a throat with extended cusp]

• One can clearly feel uneasy about our extended axionic direction: \textit{It is very different from a geodesic.}

![Diagram comparing axion and geodesic paths]
• Indeed, the distance between two maximally separated points on the longest axion-trajectory grows $\sim N$.

• By contrast, the actual (geodesic) distance grows only $\sim \ln N$.

• This is not too surprising: Our geometry is locally always that of the hyperbolic plane.

• I skip further analytical work (the paper is in the process of being written) and formulate our precise conjecture...
• Choose an $\epsilon \ll 1$. Restrict the moduli space of a given model by demanding $\Lambda/M_P > \epsilon$.

[Masses of KK or string states should not fall below $\Lambda$. This cuts off the infinite throats at a distance $\sim \ln(1/\epsilon)$.]

• **Moduli Space Size Conjecture:**
The resulting moduli space has physical diameter $\lesssim \ln(1/\epsilon)$.

This requires a number of comments....

• First, concerning distances along the throat, this is basically the Ooguri-Vafa swampland conjecture.

• Second, concerning axionic directions without flux, this is just ‘Banks et al.’

• But, including axionic directions and fluxes, this may be new and interesting, also mathematically (cf. congruence subroups and their domains).
Finally, our term ‘physical diameter’ $D$ has to be discussed.

First, as in math,

$$D \equiv \sup_{p,q} \inf_{L} \int_{L(p,q)} ds,$$

where $L(p, q)$ is a smooth curve connecting points $p$ and $q$.

But second, in contrast to the standard math definition, we allow for curves $L$ which jump from one boundary point to another.

In this way, we are sure that raising $\epsilon$ does not make the manifold larger.
Summary/Conclusions

• Axionic directions may be extended in fluxed geometries, in apparent conflict with the WGC.

• But the corresponding, appropriately defined, moduli-space distances do not grow faster than logarithmic.

• This can be formalized in a Moduli Space Size Conjecture.

• Interesting mathematical structures (fundamental domains of congruence subgroups) arise as descriptions of the relevant Flux-Extended Moduli Spaces.

• The fate of large field inflation entirely depends on effects destroying the moduli space (instantons, SUSY breaking).