Surprisingly Complex Punctures from a Dynamical System

Falk Hassler

in collaboration with

Jonathan J. Heckman

University of North Carolina at Chapel Hill

July 4, 2017
Theories of class S [Gaiotto, 2012]

6D $\mathcal{N} = (2, 0)$ SCFT

- IIB on $\mathbb{R}^{5,1} \times \mathbb{C}^2 / \Gamma$, $\Gamma \subset \text{SU}(2) \rightarrow \text{ADE-classification}$ [Witten, 1995]

- $N$ M5-branes in flat space ($A_N$) [Strominger, 1996]
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4D $\mathcal{N} = 2$ SCFT

- gauge group $G$

- flavor symmetry from punctures on $\Sigma$
Constrains on punctures

- compactify 6D $\mathcal{N} = (2, 0)$ on $S^1$

- 5D $\mathcal{N}=2$ gauge theory with matter in bifundamental of $G$
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- maximally SUSY puncture $\rightarrow$ 1/2 BPS equations for

$$\Sigma(t) = \frac{\Sigma}{t}, \quad Q(t) = \frac{Q}{t}, \quad \tilde{Q}(t) = 0$$

(in terms of $\mathcal{N}=1$ 4D superfields)

- results in Nahm pole equations [Nahm, 1980]

$$[\Sigma, Q] = Q, \quad [Q, Q^\dagger] = \Sigma$$
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- $\Sigma, Q, Q^\dagger$ are representations of $\mathfrak{su}(2)$
Generalization to $\mathcal{N}=1$?

- 6D $\mathcal{N}=(1,0)$ SCFT
- compactification $\Sigma$ with punctures $\rightarrow$ 4D $\mathcal{N}=1$ SCFTs

[Razamat, Vafa, and Zafrir, 2016]
Generalization to $\mathcal{N}=1$?

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Challenges

- much more 6D $\mathcal{N} = (1, 0)$ than $\mathcal{N} = (2, 0)$ SCFTs [Heckman, Morrison, and Vafa, 2014]
- less constrained by SUSY
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Challenges

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- use “simple” 6D $\mathcal{N}=(1,0)$ SCFT $\mathcal{N}$ M5-branes probing ADE-singularity $\mathbb{C}^2/\Gamma$
- try to classify all punctures
- harder than you might think

[Heckman, Jefferson, Rudelius, and Vafa, 2016]
[Razamat, Vafa, and Zafrir, 2016]
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Theories of Class $S_{\Gamma}$... [Heckman, Jefferson, Rudelius, and Vafa, 2016]

- stack of $N$ M5-branes probing ADE-singularity $\mathbb{C}^2/\Gamma$
- compactification on $S^1 \rightarrow 5$D quiver gauge theory
Theories of Class $S_{\Gamma}$... [Heckman, Jefferson, Rudelius, and Vafa, 2016]

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- organized according to extended Dynkin diagrams

$\hat{A}_k$

$\hat{D}_k$

$\hat{E}_6$

similar for $\hat{E}_7$ and $\hat{E}_8$
...and their punctures

- again, maximally SUSY punctures $\rightarrow$ 1/2 BPS equations for

$$\Sigma(t) = \frac{\Sigma}{t} \quad Q(t) = \frac{Q}{t} \quad \tilde{Q}(t) = \frac{\tilde{Q}}{t}$$

(in terms of $\mathcal{N}=1$ 4D superfields in covering space)
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(in terms of $\mathcal{N}=1$ 4D superfields in covering space)

- results in generalized Nahm pole equations

$$
[\Sigma, Q] = Q, \quad [Q, \tilde{Q}] = 0
$$

$$
[\Sigma, \tilde{Q}] = \tilde{Q}, \quad [Q, Q^\dagger] + [\tilde{Q}, \tilde{Q}^\dagger] = \Sigma
$$

plus invariance under $\Gamma$-action with

doublet $\left( \begin{array}{c} Q \\ \tilde{Q} \end{array} \right)$ and singlet $\Sigma$

[Heckman, Jefferson, Rudelius, and Vafa, 2016]
A closer look at $\hat{A}_k$ quivers

- choose $\Gamma \ni \gamma = \text{diag}(1_N, \omega^1 1_N, \omega^2 1_N, \ldots, \omega^k 1_N)$

$$\gamma Q \gamma^\dagger = \omega Q \quad \gamma \tilde{Q} \gamma^\dagger = \omega^{-1} \tilde{Q} \quad \gamma \Sigma \gamma^\dagger = \Sigma$$
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\gamma \Sigma \gamma^\dagger = \Sigma
$$

$$
\Sigma = \begin{pmatrix}
p(1) \\
\vdots \\
p(k)
\end{pmatrix} \\
Q = \begin{pmatrix}
q(1) \\
\vdots \\
q(k)
\end{pmatrix} \\
\tilde{Q} = \begin{pmatrix}
\tilde{q}(1) \\
\vdots \\
\tilde{q}(k-1)
\end{pmatrix}
$$

Class $S$ Class $S_{\Gamma}$ Dynamical system Summary
$N=1$ $\hat{A}_k$ quivers and a dynamical system

- rewrite gen. Nahm pole eq. in terms of $q(i)$, $\tilde{q}(i)$ and $p(i)$

\[
\begin{align*}
[Q, \tilde{Q}] &= 0 \quad \Rightarrow \quad q(i + 1)\tilde{q}(i + 1) = q(i)\tilde{q}(i) \\
[Q, Q^\dagger] + [\tilde{Q}, \tilde{Q}^\dagger] &= \Sigma \quad \Rightarrow \quad x(i) - x(i - 1) = p(i) \\
[\Sigma, Q] &= Q \quad \Rightarrow \quad q(i)\left(p(i) - p(i + 1)\right) = q(i) \\
[\Sigma, \tilde{Q}] &= \tilde{Q} \quad \Rightarrow \quad -\tilde{q}(i)\left(p(i) - p(i + 1)\right) = \tilde{q}(i)
\end{align*}
\]

with $x(i) = q(i)q(i)^* - \tilde{q}(i)\tilde{q}(i)^*$
\(N=1 \hat{A}_k\) quivers and a dynamical system

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- \(Q\) is nilpotent, thus \(Q^k = 1_k \prod_{i=1}^{k} q(i) = 0 \quad \rightarrow \quad q(i)\tilde{q}(i) = 0\)

- knowing \(x(i)\) is sufficient to get \(q(i)\) and \(\tilde{q}(i)\)
\( N=1 \, \hat{A}_k \) quivers and a dynamical system

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- Discrete dynamical system

\[
f : \begin{pmatrix} p \\ x \end{pmatrix}(i + 1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}(i) - \text{sgn} \, x(i)
\]
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\]

- choose \( x(1), p(1) \) and all other \( x(i), p(i) \) are fixed

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\(^1\) In general \( p(i + 1) \) is unconstrained if \( x(i) = 0 \). We choose \( p(i + 1) = p(i) \) to formally extend the dynamical system beyond this point.
Periodic orbits

- punctures = periodic orbits of length $k = |\Gamma|$
- strongly depends on the initial condition, e.g.
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$$k = 100, \quad x(1) = -\frac{48}{15}, \quad p(1) = -\frac{49}{15}$$
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- How to find the right initial conditions?
A tree of solutions

- periodic orbits of type $x(k) = 0$ organized in tree structure
A tree of solutions

- periodic orbits of type \( x(k) = 0 \) organized in tree structure

\[ \text{Class S} \]

\[ \text{Class } S_T \]

\[ \text{Dynamical system} \]

\[ \text{Summary} \]
and qualitative

- the tree of solutions is surprisingly complex
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- any pattern? e.g. self similar like Barnsley’s fern?
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- any pattern? e.g. self similar like Barnsley’s fern?
- even # of solutions has interesting structure
Summary

Even for the simplest class $\mathcal{S}_\Gamma$ theories, the punctures show an amazingly rich structure compared to the $\mathcal{N} = 2$ case.

still lots of questions

- quantitative measure for complexity
- connection to spin chain
- statistical properties of solutions
- are the characteristic quantities for a puncture
- can we do more for $N > 1$, e.g. large $N$ limit AdS/CFT