Calabi-Yau Fourfolds with non-trivial Three-Form Cohomology

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String Pheno 2017
Motivation

F-theory effective action on elliptically fibered Calabi-Yau fourfold $Y^4$:

$\mathcal{N} = 1$ gauged supergravity in (3 + 1) dimensions

$S(4) = \int_{M^3} R(4) - K F \bar{F} D M^I \wedge \ast D \bar{M}^J - \frac{1}{2} \text{Re}(f) \Lambda \Sigma F \Lambda \wedge \ast F \Sigma - \frac{1}{2} \text{Im}(f) \Lambda \Sigma F \Lambda \wedge F \Sigma$

Massless spectrum includes complex scalars from non-trivial harmonic three-forms of $Y^4$.

⇒ What can we learn about their dynamics?

⇒ Can we construct explicit examples?

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\[
S^{(4)} = \int_{\mathcal{M}_{3,1}} \frac{1}{2} R^{(4)} \ast 1 - K_{i \bar{j}} \mathcal{D} M^i \wedge \ast \mathcal{D} \bar{M}^j
- \frac{1}{2} \text{Re} (f)_{\Lambda \Sigma} F^{\Lambda} \wedge \ast F^{\Sigma} - \frac{1}{2} \text{Im} (f)_{\Lambda \Sigma} F^{\Lambda} \wedge F^{\Sigma}
\]

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$\Rightarrow$ Can we construct explicit examples?
Outline

Three-Forms on Calabi-Yau Fourfolds

Calabi-Yau Fourfolds as Toric Hypersurfaces

Metric for Three-Form Moduli of a Toric Hypersurface

Outlook
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- Three-Forms on Calabi-Yau Fourfolds
- Calabi-Yau Fourfolds as Toric Hypersurfaces
- Metric for Three-Form Moduli of a Toric Hypersurface
- Outlook
Geometry and Topology of $CY_4$

Characteristic feature: $SU(4)$-holonomy

Covariantly constant Weyl-spinor: $\nabla \eta = 0$, $\gamma^9 \eta = \eta \Rightarrow N = 2$ 3d $M$-theory vacua!

$\Rightarrow N = 1$ 4d $F$-theory vacua!

(no flux)

Hodge diamond:

$$
\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & h_1 & 0 & h_3 \\
0 & 0 & h_1 & 0 & 0 & h_2 & 2 & h_3 \\
h_2 & 1 & h_2 & 1 & 0 & h_2 & 0 & h_1 \\
0 & 0 & h_1 & 0 & 0 & h_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

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Blacksburg, Juli 2017
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$$\Rightarrow \mathcal{N} = 1 \text{ 4d F-theory vacua!}$$
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Hodge diamond:

$$
\begin{array}{cccccc}
    & & & 1 & & \\
    & & 0 & 0 & & \\
    & 0 & h^{1,1} & 0 & & \\
 1 & h^{2,1} & h^{2,1} & 0 & & \\
     & h^{3,1} & h^{2,2} & h^{3,1} & 1 & \\
    0 & h^{2,1} & h^{2,1} & 0 & & \\
    0 & h^{1,1} & 0 & & & \\
    0 & 0 & 1 & & & \\
\end{array}
$$
Expansion of the Three-Form

Consider scalar three-form moduli arising from \( C_3 \in H^{2,1}(Y_4) \oplus H^{1,2}(Y_4) \)

Kinetic term: \( dC_3 \wedge *dC_3 \Rightarrow \) Hodge-star depends on moduli!

precisely:
\[
*\psi = iJ \wedge \psi, \quad \psi \in H^{2,1}(Y_4), \quad J \in H^{1,1}(Y_4)
\]

\( \Rightarrow \) Choose three-form basis \( \Psi^l \) depending on complex structure moduli!

\[
\Psi^l(z, \bar{z}) = \frac{1}{2} \text{Re}(f)^{lm}(\alpha_m - i\bar{f}_{mk}\beta^k) \in H^{1,2}(Y_4)
\]

Three-form deformations:

\[
C_3 = \sum_{l=1}^{h^{2,1}} N_l \Psi^l + \text{c.c.}
\]

\( \Rightarrow \) \( h^{2,1} \) complex scalars \( N_l \)

\( \Rightarrow \) bosonic part of chiral multiplets
Three-Form Ansatz

Suggested by [Grimm '10]: 
\((l, k, m = 1, \ldots, h^{2,1})\)

\[
\psi^l = \frac{1}{2} \text{Re}(f)^{\text{lm}} (\alpha_m - i f_{mk} \beta^k) \in H^{1,2}(Y_4)
\]

Properties:
- \(\alpha_l, \beta^k\) basis of \(H^3(Y_4, \mathbb{R})\) \(\Rightarrow\) topological
- \(f_{lm}(z)\) holomorphic (three-form periods)
- Assume: \(\beta_l \wedge \beta_m = 0\)

Result:
\[
\frac{\partial \psi^l}{\partial z^K} = \frac{\partial \psi^l}{\partial z^K} \sim \psi^m
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Advantage:
\[M_{A_l}^k = \int \omega_A \wedge \alpha_l \wedge \beta^k\] topological intersection numbers

\[\Rightarrow \int_{Y_4} \psi^l \wedge \ast \bar{\psi}^k = -\frac{1}{2} \text{Re}(f)^{lm} M_{Am}^k v^A\]

Result:
\[\frac{\partial \psi^l}{\partial z^K} = \frac{\partial \bar{\psi}^l}{\partial z^K} \sim \psi^m\]

Goal of our work:
Calculate \(f\) and \(M\)

\[K_{new}^M \sim \text{Re}(N)_l \text{Re}(f)^{lm} M_{Am}^k v^A \text{Re}(N)_k \Rightarrow \text{Im}(N)_l\] axionic!
Now Calabi-Yau fourfold \( Y_4 \) smooth toric hypersurface Toric divisors \( D_i \), codimension one submanifolds of \( Y_4 \) invariant under torus action

Non-trivial three- and two-forms from divisors \( D_i \) via Gysin-morphism of their inclusion \( \iota_i: D_i \rightarrow Y_4 \) [Danilov; Mavlyutov]

Sebastian Greiner (MPP Munich)  Three-forms of CY_4  Blacksburg, Juli 2017  \( \text{7 } / \text{14} \)
Now Calabi-Yau fourfold $Y_4$ smooth toric hypersurface

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Non-trivial three- and two-forms from divisors $D_i$ via Gysin-morphism of their inclusion

$$\nu_i : D_i \rightarrow Y_4 \quad [\text{Danilov;Mavlyutov}]$$
Gysin-Morphism

\[ \iota \_\star : H^n(D_i, \mathbb{C}) \xrightarrow{PD} H_{6-n}(D_i, \mathbb{C}) \xrightarrow{(\iota_\star)^{-1}} H_{6-n}(Y_4, \mathbb{C}) \xrightarrow{PD} H^{n+2}(Y_4, \mathbb{C}) \]

⇒ Respect Hodge decomposition:

\[ \iota \_\star : H^{0,0}(D_i) \longrightarrow H^{1,1}(Y_4), \]
\[ \iota \_\star : H^{1,0}(D_i) \longrightarrow H^{2,1}(Y_4). \]

For smooth toric Calabi-Yau hypersurfaces \( Y_4 \)

\[ \bigoplus_i H^{0,0}(D_i) \cong H^{1,1}(Y_4), \]
\[ \bigoplus_i H^{1,0}(D_i) \cong H^{2,1}(Y_4), \bigoplus_i H^{0,1}(D_i) \cong H^{1,2}(Y_4). \]

⇒ Construct \( H^{1,0}(D_i) \) for toric divisors \( D_i \) of \( Y_4 \)!
Divisors with non-trivial One-Forms

\[ H^{1,0}(D) \simeq H^{1,0}(R) \]

For \( H^{1,0}(D) \neq 0 \):

- Toric divisor \( D \) fibration
- Fiber: \( E \) toric surface,
  \[ (h^i,j(E) = 0 \text{ for } i \neq j) \]
- Base: \( R \) Riemann surface
Construct three-forms:

\[ \psi_A = \iota_{l \ast} (\gamma_a) \in H^{2,1}(Y_4) \]

with \( A = (l, a) \)

\[ \iota_l : D_l \rightarrow Y_4, \quad \gamma_a \in H^{1,0}(R) \]

\[ \Rightarrow \text{Complex structure dependence of } \psi_A \text{ determined by } H^{1,0}(R) \]
Three-forms of Calabi-Yau Fourfold Hypersurface

Construct three-forms:

\[ \psi_A = \iota_l^* (\gamma_a) \in H^{2,1}(Y_4) \]

with \( A = (l, a) \)

\[ \iota_l : D_l \to Y_4, \quad \gamma_a \in H^{1,0}(R) \]

\( \Rightarrow \) Complex structure dependence of \( \psi_A \) determined by \( H^{1,0}(R) \)

Normalized basis: \( \gamma_a = \alpha_a + i f_{ab}(z) \beta^b \in H^{1,0}(R), \quad \alpha_a, \beta^b \in H^1(R, \mathbb{Z}) \)

Metric on \( H^{1,0}(R) \):

\[ -i \int_R \gamma_a \wedge \bar{\gamma}_b = 2 \cdot \text{Re}(f)_{ab} > 0, \quad f_{ab} = f_{ba} \]
Recall metric on $H^{2,1}(Y_4)$

$$Q_{AB} = \int_{Y_4} \psi_A \wedge \ast \bar{\psi}_B = -i \int_{Y_4} J \wedge \psi_A \wedge \bar{\psi}_B, \quad J \in H^{1,1}(Y_4)$$

Two- and three-forms from toric divisors via Gysin-map

$\Rightarrow$ Triple intersection of toric divisors!

Only divisors that fiber over same $R$ contribute!

$$D_I \cap D_m \cap D_p = M_{Imp} \cdot R$$

with $M_{Imp} = \#E_I \cap E_m \cap E_p$

(Generalized sphere tree)
Three-Form Metric for Fourfold Hypersurface

Metric on $H^{2,1}(Y_4)$

$$Q_{AB} = 2v^l M_{mp} \text{Re}(f)_{ab}$$

$A = (m, a), \ B = (p, b)$

Reorder indices to find previous result

$$Q_{AB} = 2v^\Sigma M_{\Sigma A}^C \text{Re}(f)_{CB}$$

$v^\Sigma M_{\Sigma A}^C$ linear dependence on Kähler moduli $v^\Sigma$

$\text{Re}(f)_{CB}$ complicated dependence on complex structure moduli $z^K$

$\Rightarrow$ Toric data exchanged by mirror symmetry!
Outlook

Next step:

Discuss F-theory physics on elliptically fibered $CY_4$ and their weak coupling limit on $CY_3$

Construct examples with three-forms such that weakly coupled IIB has

- Wilson line moduli on a $D7$ wrapping a four-cycle of $CY_3$ (axions)
- Odd moduli, scalars from $B_2, C_2$ (axions)

⇒ Metric on $H^{2,1}(Y_4)$ determines decay-constants!
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$\Rightarrow$ Metric on $H^{2,1}(Y_4)$ determines decay-constants!

Future directions:

- Construct bases with three-forms (bulk U(1))
- Generalize to complete intersections
- Include $G_4$-flux and calculate superpotential
- ...

Thank you for your attention! Questions?