Heterotic Flux Compactifications

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String Pheno 2017
Virginia Tech, 7 July 2017

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Again???

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This talk is about the internal geometry of four-dimensional $N = 1$ compactifications of heterotic string theory with flux $H$.

Hull '86, Strominger '86

For Calabi-Yau compactifications ($H = 0$, $\phi$ constant), topology of internal space determines the field content of the 4D-supergravity.

$h^{1,1} \sim$ sizes of 2-cycles, $h^{2,1} \sim$ sizes of 3-cycles.

Candelas-Horowitz-Strominger-Witten '85

Furthermore, Yau’s Theorem ’76 allowed the application of powerful methods from algebraic and Kähler geometry leading to important advances.

However, the many massless scalar fields arising from generic CY compactifications called for stabilising mechanisms.
Outline

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However, the many massless scalar fields arising from generic CY compactifications called for stabilising mechanisms.
In the presence of fluxes, the geometry is not Kähler and classical methods no longer apply. Furthermore, $H \neq 0$ requires holomorphic bundles which make the analysis really complicated

$$dH = \alpha'(\text{tr} R \wedge R - \text{tr} F \wedge F).$$

Recent developments in the study of

- geometry of equations of motion (GF '13, Coimbra-Minasian-Triendl-Waldran '14)
- heterotic T-duality (Baraglia-Hekmati '13, GF '16)
- moduli (Melnikov-Sharpe '11; De la Ossa-Svanes '14; Anderson-Gray-Sharpe '14; GF-Rubio-Tipler '15)

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Moduli
In 1986 Hull and Strominger characterized warped 4d compactifications of the heterotic string, with $N = 1$ supersymmetry, non-zero flux $H \neq 0$, and non-constant dilaton $\phi$.

Given by an $SU(3)$-structure $(\psi, \omega)$ with almost complex structure $J: TM^6 \to TM^6$ and metric $g$, and a gauge field $A$ with field strength $F$, such that

\begin{align*}
    d\Omega &= 0, \\
    g^{i\bar{j}} F_{i\bar{j}} &= 0, \\
    F_{i\bar{j}} &= 0, \\
    d^* \omega - i(\overline{\partial} - \partial) \log \|\Omega\| &= 0, \\
    2i \partial \overline{\partial} \omega - \alpha'(\text{tr} R \wedge R - \text{tr} F \wedge F) &= 0,
\end{align*}

where $\Omega = e^\phi \psi$, $H = i(\overline{\partial} - \partial)\omega$, $\phi = \log \|\Omega\|

In this talk, first order equations in $\alpha'$-expansion taken as exact (my apologies). Imposing the equations of motion, is equivalent to

\begin{align*}
    g^{i\bar{j}} R_{i\bar{j}} &= 0, \\
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Reminiscent of $R = R_{\nabla^\omega}$ in $\alpha'$-expansion.
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g^{ij}R_{ij} = 0, \quad R_{ij} = 0.
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Reminiscent of \( R = R_\nabla \) in \( \alpha' \)-expansion.
**Theorem (____-Rubio-Tipler ’15)**

The Hull-Strominger system (below) is elliptic. Therefore, the moduli space is finite-dimensional.

\[ d\Omega = 0, \]
\[ g^{ij} F_{ij} = 0, \quad F_{ij} = 0, \]
\[ g^{ij} R_{ij} = 0, \quad R_{ij} = 0, \]
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**Remark:** does not take into account \( b \)-fields. Where are they?

Canonically attached to \( M \) and the gauge bundle \( V \), there is a set of string classes \( H^3_{str}(V, \mathbb{R}) \) (Redden ’11). Fixing \( (H_0, \nabla_0, A_0) \) a solution of the Bianchi identity, yields identification

\[ H^3_{str}(V, \mathbb{R}) \cong H^3(M, \mathbb{R}) \]

\[ dH_0 = \alpha' \text{tr } R_{\nabla_0} \wedge R_{\nabla_0} - \alpha' \text{tr } F_{A_0} \wedge F_{A_0}. \]
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Flux map

A choice of integral string class $\sigma \in H^3_{str}(V, \mathbb{Z})$, corresponds to a string structure (up to homotopy) Killingback '87, Witten '87 $\sim$ choice of quantization scheme for heterotic $\Sigma$-model Freed '86, Waldorf '13.

There is a well-defined map flux on moduli, so that (a first approximation to) moduli of heterotic compactifications is given by $M_{HS}^\sigma = \text{flux}^{-1}(\sigma)$

\[ M_{HS} \xrightarrow{\text{flux}} H^3_{str}(V, \mathbb{R}). \]

**Theorem (____-Rubio-Tipler '15)**

$M_{HS}^\sigma$ corresponds to a moduli space of natural Killing spinor equations in generalized geometry, on a Courant algebroid $E_\sigma$ determined by $\sigma \in H^3_{str}(V, \mathbb{Z})$.

**Idea:** infinitesimally

$\delta\text{flux}(\dot{\Omega}, \dot{\omega}, \dot{\nabla}, \dot{A}) = [\delta(d_{\text{f}}^\alpha \omega) - 2\alpha' \text{ tr } \dot{\nabla} \wedge R_{\nabla} + 2\alpha' \text{ tr } \dot{A} \wedge F_A] \in H^3(M, \mathbb{R})$. 
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$M_{HS}^σ$ corresponds to a moduli space of natural Killing spinor equations in generalized geometry, on a Courant algebroid $E_σ$ determined by $σ ∈ H^3_{str}(V, ℤ)$.

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$$\delta \text{flux}(\dot{Ω}, \dot{ω}, \dot{∇}, \dot{A}) = [δ(d^c_jω) - 2α' tr \dot{∇} ∧ R_∇ + 2α' tr \dot{A} ∧ F_A] ∈ H^3(M, ℜ).$$

Now, $δ \text{flux} = 0$, implies there exists a 2-form $b$ on $M$ such that

$$db = δ(d^c_jω) - 2α' tr \dot{∇} ∧ R_∇ + 2α' tr \dot{A} ∧ F_A.$$

- $(ω, b, ∇, A)$ determines generalized metric on $E_σ = T ⊕ T^* ⊕ \text{End} T ⊕ \text{End} V$

- Having added parameters $b ∈ Ω^2(M)$, need to add symmetries: $B$-field symmetries.
- Imposing equations are preserved by $B$-field symmetries: automorphism group of $E_σ$. 
Theorem (_____—Rubio-Tipler ’15)

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$\delta \text{flux}(\dot{\Omega}, \dot{\omega}, \dot{\nabla}, \dot{A}) = [\delta(d^c J \omega) - 2\alpha' \text{tr} \dot{\nabla} \wedge R_{\nabla} + 2\alpha' \text{tr} \dot{A} \wedge F_A] \in H^3(M, \mathbb{R})$.

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\( M^\sigma_{HS} \) corresponds to a moduli space of natural Killing spinor equations in generalized geometry, on a Courant algebroid \( E_\sigma \) determined by \( \sigma \in H^3_{str}(V, \mathbb{Z}) \).

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Kähler moduli

Fixing the complex structure $X$ on $M$, and consider the Aeppli cohomology group

$$H^{1,1}_A(X) = \frac{\text{Ker } i\partial\bar{\partial}: \Omega^{1,1} \to \Omega^{2,2}}{\text{Im } \partial \oplus \bar{\partial}: \Omega^{0,1} \oplus \Omega^{1,0} \to \Omega^{1,1}}.$$ 

Fixing the holomorphic bundle structure on $V$ and $TX$, the equation

$$db = \delta(d\dot{\omega}) - 2\alpha' \text{tr } \dot{\nabla} \wedge \mathcal{R} + 2\alpha' \text{tr } \dot{A} \wedge F_A$$

implies that there is a well-defined map

$$[(\dot{\Omega}, \dot{\omega}, \dot{\nabla}, \dot{A})] \to [\dot{\omega} + ib - 2\alpha' \text{tr } s_1 \wedge \mathcal{R} + 2\alpha' \text{tr } s_2 \wedge F_A] \in H^{1,1}_A(X)$$

for suitable infinitesimal complex gauge transformations $s_1, s_2$. 

Kähler moduli of a heterotic flux compactification is $H^{1,1}_A(X)$

($\sim d$-closed $(1,1)$-currents $\supset$ analytic 2-cycles)

Remark: assuming $\partial\bar{\partial}$-Lemma holds, De la Ossa-Svanes '14 and Anderson-Gray-Sharpe '14 propose $H^{1,1}_\partial(X)$ as Kähler moduli.
Kähler moduli

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Fixing the holomorphic bundle structure on $V$ and $TX$, the equation

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$$db = \delta(d\tilde{f}\omega) - 2\alpha' \text{tr} \hat{\nabla} \wedge R_V + 2\alpha' \text{tr} \hat{A} \wedge F_A$$

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$$[(\hat{\Omega}, \hat{\omega}, \hat{\nabla}, \hat{A})] \to [\hat{\omega} + ib - 2\alpha' \text{tr} s_1 \wedge R_V + 2\alpha' \text{tr} s_2 \wedge F_A] \in H_{\partial\bar{\partial}}^{1,1}(X)$$

for suitable infinitesimal complex gauge transformations $s_1, s_2$.

Kähler moduli of a heterotic flux compactification is $H_{\partial\bar{\partial}}^{1,1}(X)$
($\sim d$-closed $(1,1)$-currents ⊃ analytic 2-cycles)

**Remark:** assuming $\partial\bar{\partial}$-Lemma holds, De la Ossa-Svanes ’14 and Anderson-Gray-Sharpe ’14 propose $H_{\partial}^{1,1}(X)$ as Kähler moduli.
Twisted Heterotic Compactifications
A key ingredient for the description of Hull-Strominger system using generalized geometry in arXiv:1611.08926 is the treatment of the dilaton field $\phi$ using Dirac generating operators (Ševera, Alekseev-Xu, unpublished '01). In this approach, $\phi$ may be only locally defined, with field-strength given by a closed $1$-form in the internal manifold $M$.

‘Assuming $H^1(M)$ is trivial, this implies ... a globally defined scalar field $\phi$. If $H^1(M)$ is not trivial, there are interesting new possibilities [30].’

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Common sector: A twisted heterotic compactification to $10 - 2n$ dimensions is given by a $SU(n)$-structure $(\omega, \Psi)$ on $M^{2n}$ satisfying

\[
\begin{align*}
  d\Psi - \theta \wedge \Psi &= 0, \\
  d\theta &= 0, \\
  dd^c \omega &= 0.
\end{align*}
\]

(1)

where $\theta = Jd^* \omega$ is the Lee form of the Hermitian structure.

- Mild class of non-geometric backgrounds, which violate Maldacena-Nuñez Theorem.
- Very small moduli space.

Remark: If $M$ is compact and $\theta$ exact, then $H = 0$, $\theta = 0$ and $g$ is a Calabi-Yau metric.
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**Example:** Twisted compactification to 6d on $M^4 = S^3 \times S^1$.

Identify $M \cong SU(2) \times U(1)$, with Lie algebra

$$\mathfrak{su}(2) \oplus \mathbb{R} = \langle e_1, e_2, e_3, e_4 \rangle,$$

where

$$de^1 = e^2 \wedge e^3, \quad de^2 = e^3 \wedge e^1, \quad de^3 = e^1 \wedge e^2, \quad de^4 = 0.$$

Using T-duality with parameters, obtain all homogeneous solutions on $M$, given by $(\beta, \gamma, \ell, \tau) \in \mathbb{R}_{>0} \times \mathbb{R}^3$.

$$g = e^1' \otimes e^1' + e^2 \otimes e^2 + e^3 \otimes e^3 + \frac{\gamma^2}{\beta^2} e^4 \otimes e^4$$

$$H = -\beta e^1' \wedge de^1', \quad \Psi = (e^1' + i\frac{\gamma}{\beta} e^4) \wedge (e^2 + ie^3)$$

$$\theta = \frac{\gamma}{\beta} e^4, \quad b = \tau e^1' \wedge e^4$$

where $e^1' = e^1 + \frac{\ell}{\beta} e^4$.

**Remark:** These are homogeneous primary Hopf surfaces. Moduli of homogeneous complex structures $M_{\text{cx}} = \mathbb{C}$ (Hasegawa-Kamishima).
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Moduli $\mathcal{M}$ (twisted common sector)

\[(\beta, \gamma, \ell, \tau) \in \mathcal{M} \quad \xrightarrow{\text{flux}} \quad H^3(M, \mathbb{R}) = \mathbb{R} \quad \exists \beta, \quad \frac{\gamma}{\ell} + i\frac{\beta}{\ell} \in \mathcal{M}_{cx} = \mathbb{C} \quad \exists \gamma + i\tau, \quad H^1_A(X) = \mathbb{C} \]

Remarks:

- $\partial\bar{\partial}$-Lemma does not hold (in fact, $H^1_\partial(X) = 0$). Kähler moduli given by non-topological, transcendental quantities.
- Standard compactifications on $K3$ have (real) 82-dimensional moduli.
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Heterotic T-duality
Fix a heterotic flux compactification \((\Omega, \omega, \nabla, A)\) with gauge group \(G\), let 
\[ P = P_g \times_M P_V \] (a \(\text{Spin}(6, \mathbb{R}) \times G\)-bundle).

**Definition**: The string class of the heterotic flux compactification 
\((\Omega, \omega, \nabla, A)\) is

\[
[\hat{H}] = [p^* d^c \omega - \alpha' \text{CS}(\nabla) + \alpha' \text{CS}(A)] \in H^3(P, \mathbb{R}),
\]

where

\[
\text{CS}(A) = \text{tr}(A \wedge F_A + \frac{1}{6} A \wedge [A \wedge A])
\]

Assuming that \(M\) and \(V\) have torus action \(T^k\), can consider \(T^k\)-invariant string classes:

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Let \((P, \sigma), (\overline{P}, \overline{\sigma})\) a pair as before, with commuting diagram \((B = M/T^k = \overline{M}/\overline{T}^k \text{ and } P_0 = P/T^k = \overline{P}/\overline{T}^k)\).

\[
\begin{array}{ccc}
P \times_{P_0} \overline{P} & \xrightarrow{p} & \overline{P} \\
P & \xrightarrow{p} & P_0 & \xleftarrow{\overline{p}} & \overline{P} \\
M & \xrightarrow{} & B & \xleftarrow{} & \overline{M}
\end{array}
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**Definition** (Baraglia-Hekmati '15): \((P, \sigma)\) is T-dual to \((\overline{P}, \overline{\sigma})\) if exists an invariant 2-form \(F\) on \(P \times_{P_0} \overline{P}\) which is ‘non-degenerate on the fibres’ of \(P \times_{P_0} \overline{P} \to P_0\) and representants \(\hat{\mathcal{H}} \in \sigma\) and \(\hat{\overline{\mathcal{H}}} \in \overline{\sigma}\), such that \(p^* \hat{\mathcal{H}} - \overline{p}^* \hat{\overline{\mathcal{H}}} = dF\).

**Theorem (GF '16)**

Solutions of the Hull-Strominger system are preserved by Heterotic T-duality.

**Remarks:** Existence of \(T\)-dual requires flux quantization \(\sigma \in H^3_{\text{str}}(V, \mathbb{Z})\). Previous partial check by (Evslin-Minasian '08).
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Thank you!