Zero Mode Counting in F-Theory via CAP

Martin Bies

String Pheno 2017
Overview

Task

4 dim. F-theory compactification

Count (anti)-chiral massless matter fields in 4d effective theory
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4 dim. F-theory compactification → Count (anti)-chiral massless matter fields in 4d effective theory

Structure

- Analyse physics (C. Mayrhofer, T. Weigand, M.B. – 1706.04616)
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| 4 dim. F-theory compactification | Count (anti)-chiral massless matter fields in 4d effective theory |

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- Analyse physics (C. Mayrhofer, T. Weigand, M.B. – 1706.04616)
  - Compute sheaf cohomologies of **non-pullback** line bundles
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### Task

| 4 dim. F-theory compactification | Count (anti)-chiral massless matter fields in 4d effective theory |

### Structure

- Analyse physics (C. Mayrhofer, T. Weigand, M.B. – 1706.04616)
  - Compute sheaf cohomologies of **non-pullback** line bundles
- Developed and implemented algorithms with M. Barakat et al. ([GitHub](https://github.com/homalg-project/CAP_project) – 1003.1943, 1202.3337, 1210.1425, 1212.4068, 1409.2028, 1409.6100)
Schematic Picture: Physics and Geometry of F-theory

- fibre
- base $B_3$

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a-th state in rep. $R \leftrightarrow \text{matter surface} \quad S_R^a = \sum_{i=1}^{n} a_i \mathbb{P}_i (C_R)$
Counting zero modes

From Physics of F-Theory to Line Bundles

- a-th state in rep. $\mathbf{R} \leftrightarrow$ matter surface $S_{\mathbf{R}}^a = \sum_{i=1}^{n} a_i \mathbb{P}_i^1 (C_{\mathbf{R}})$
- $G_4$-flux $\leftrightarrow$ (complex) 2-cycle $A$ in $Y_4$

Chiral zero modes of $S_{\mathbf{R}}^a \leftrightarrow H^0 (\mathcal{C}_{\mathbf{R}}, L(S_{\mathbf{R}}^a, A) \otimes O_{\mathcal{C}_{\mathbf{R}}})$

Anti-chiral zero modes of $S_{\mathbf{R}}^a \leftrightarrow H^1 (\mathcal{C}_{\mathbf{R}}, L(S_{\mathbf{R}}^a, A) \otimes O_{\mathcal{C}_{\mathbf{R}}})$
## Counting zero modes

### From Physics of F-Theory to Line Bundles

*a*-th state in rep. $\mathbf{R} \leftrightarrow$ matter surface $S^a_{\mathbf{R}} = \sum_{i=1}^{n} a_i \mathbb{P}_i(C_{\mathbf{R}})$

$G_4$-flux $\leftrightarrow$ (complex) 2-cycle $A$ in $Y_4$

### Consequence

- $S^a_{\mathbf{R}}$ and $A$ intersect in number of points in $Y_4$
## Counting zero modes

### From Physics of F-Theory to Line Bundles

- $a$-th state in rep. $R \leftrightarrow$ matter surface $S^a_R = \sum_{i=1}^{n} a_i \mathbb{P}^1_i (C_R)$

- $G_4$-flux $\leftrightarrow$ (complex) 2-cycle $A$ in $Y_4$

### Consequence

- $S^a_R$ and $A$ intersect in number of points in $Y_4$

  $\Rightarrow \pi_* (S^a_R \cdot A) \leftrightarrow$ number of points in $C_R$
Counting zero modes

From Physics of F-Theory to Line Bundles

\[ S^a_R = \sum_{i=1}^{n} a_i \mathbb{P}^1_i (C_R) \]

G4-flux ↔ (complex) 2-cycle \( A \) in \( Y_4 \)

Consequence

- \( S^a_R \) and \( A \) intersect in number of points in \( Y_4 \)
- \( \pi_* (S^a_R \cdot A) \leftrightarrow \) number of points in \( C_R \)
- \( L (S^a_R, A) := O_{C_R} (\pi_*(S^a_R \cdot A)) \in \text{Pic} (C_R) \)
Counting zero modes

From Physics of F-Theory to Line Bundles

\[ a\text{-th state in rep. } R \leftrightarrow \text{matter surface } S_R^a = \sum_{i=1}^{n} a_i \mathbb{P}_i^1 (C_R) \]

\[ G_4\text{-flux } \leftrightarrow (\text{complex} )\text{ 2-cycle } A \text{ in } Y_4 \]

Consequence

- \( S_R^a \) and \( A \) intersect in number of points in \( Y_4 \)

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Zero Modes and Sheaf Cohomology

- chiral zero modes of \( S_R^a \leftrightarrow H^0 (C_R, L (S_R^a, A) \otimes O_{\text{spin}, C_R}) \)

- anti-chiral zero modes of \( S_R^a \leftrightarrow H^1 (C_R, L (S_R^a, A) \otimes O_{\text{spin}, C_R}) \)
Generalities Of The Implementations In CAP

Why new algorithm?

Let \( (S, A) = \mathcal{O}(D) \). Extend \( L(S, A) \) by zero outside of \( \mathcal{C}(R) \).

⇒ Coherent sheaf on \( X \).

Schematic Picture

Idea of mathematician G. Smith et al. (math/9807170, math/0305214, DOI: 10.4171/OWR/2013/25)

cohomCalg by R. Blumenhagen et al. (1003.5217, 1006.0780, 1006.2392, 1010.3717)

Combine to obtain algorithm which applies on more general toric spaces (than idea of G. Smith) to all coherent sheaves (i.e. not 'only' line bundles as cohomCalg).
**Generalities Of The Implementations In CAP**

<table>
<thead>
<tr>
<th>Why new algorithm?</th>
</tr>
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<tbody>
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<td>( L(S_R^a, A) = \mathcal{O}_{C_R}(D) ). Extend ( L(S_R^a, A) ) by zero outside of ( C_R ).</td>
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How to encode $\mathcal{O}_{X_\Sigma}(-D)$?
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How to encode \( \mathcal{O}_{\Sigma}(\mathcal{O}) \)?

- \(\Sigma\) toric variety (without torus factor) with coordinate ring \( S \)
- divisor \( D = V(P_1, \ldots, P_n) \) cut out by hom. polynomials
  \[ A := \ker(P_1, \ldots, P_n) \leftrightarrow \text{relations among the } P_i \]
How to encode $\mathcal{O}_{X_\Sigma}(-D)$?

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$\Rightarrow$ $A := \ker (P_1, \ldots, P_n) \leftrightarrow$ relations among the $P_i$

$\Rightarrow$ Look at exact sequence

$$
\bigoplus_{j=1}^{R_2} S(e_j) \xrightarrow{A} \bigoplus_{i=1}^{R_1} S(d_i) \twoheadrightarrow M \to 0,
$$

which defines $M \in S\text{-fpgrmod}$

$\widetilde{\mathcal{M}} \sim \mathcal{O}_{X_\Sigma}(-D)$ via the sheafification functor $\widetilde{\cdot}$:

$\mathcal{S} \text{-fpgrmod} \to \text{Coh} X_\Sigma, N \mapsto \widetilde{N}$

⇒ Use $M \in S\text{-fpgrmod}$ as computer model for $\mathcal{O}_{X_\Sigma}(-D)$
How to encode \( \mathcal{O}_{X_{\Sigma}}(-D) \)?

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Answer

- \( \tilde{M} \cong \mathcal{O}_{X_{\Sigma}}(-D) \) via the sheafification functor

\[ \tilde{\cdot} : S\text{-fpgrmod} \to \text{Coh}\ X_{\Sigma}, \ N \mapsto \tilde{N} \]
How to encode $\mathcal{O}_{X_\Sigma}(-D)$?

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$\Rightarrow$ Use $M \in S\text{-fpgrmod}$ as computer model for $\mathcal{O}_{X_\Sigma}(-D)$
Sketch of Algorithm in CAP

1. Use \text{cohomCalg} to compute \( V_k(X_\Sigma) := \{ L \in \text{Pic}(X_\Sigma), h^k(X_\Sigma, L) = 0 \} \)

2. Find ideal \( I \subseteq S \) (along idea of G. Smith) s.t. \( H^i(X_\Sigma, \tilde{M}) \sim \text{Ext}^i_S(I, M)_0 \)

3. Compute \( \text{Q-dimension of Ext}^i_S(I, M)_0 \)
Sketch of Algorithm in CAP

Input and Output

(smooth, complete) or (simplicial, projective) toric variety $X_\Sigma$

$M \in S$-fpgrmod

$h^i \left( X_\Sigma, \tilde{M} \right)$
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Step-by-step

1. Use $\text{cohomCalc}$ to compute $(0 \leq k \leq \dim_{\mathbb{Q}} (X_\Sigma))$

$$V^k (X_\Sigma) := \left\{ L \in \text{Pic} (X_\Sigma) \mid h^k (X_\Sigma, L) = 0 \right\}$$
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Input and Output

- \( C_{5-2} \subseteq \mathbb{P}^2_Q \)
- \( L_{5-2} \leftrightarrow M \) and \( M \) defined by
  - \( S(-36) \oplus S(-39) \oplus S(-41) \oplus S(-23) \oplus S(-38) \to S(-6) \oplus S(-21) \to M \to 0 \)

\[ h^1 \left( \mathbb{P}^2_Q, \tilde{M} \right) = ? \]
**Input and Output**

- \( C_{5^{-2}} \subseteq \mathbb{P}^2_Q \)
- \( L_{5^{-2}} \leftrightarrow M \) and \( M \) defined by

\[
S(-36) \oplus S(-39) \oplus S(-41) \oplus S(-23) \oplus S(-38) \rightarrow S(-6) \oplus S(-21) \rightarrow M \rightarrow 0
\]

**Apply Algorithm**

1. **Compute vanishing sets via cohomCalg:**

\[
V^0(\mathbb{P}^2_Q) = (-\infty, -1]_\mathbb{Z}, \quad V^1(\mathbb{P}^2_Q) = \mathbb{Z}, \quad V^2(\mathbb{P}^2_Q) = [-2, \infty)_\mathbb{Z}
\]
**Input and Output**

- \( C_{5_{-2}} \subseteq \mathbb{P}^2_Q \)
- \( L_{5_{-2}} \leftrightarrow M \) and \( M \) defined by
  \[
  S(-36) \oplus S(-39) \oplus S(-41) \oplus \]
  \[
  S(-23) \oplus S(-38) \rightarrow \]
  \[
  S(-6) \oplus S(-21) \rightarrow M \rightarrow 0
  \]

\[ h^1\left(\mathbb{P}^2_Q, \tilde{M}\right) = ? \]

**Apply Algorithm**

1. \( V^0(\mathbb{P}^2_Q) = (-\infty, -1][\mathbb{Z}\), \( V^1(\mathbb{P}^2_Q) = \mathbb{Z}\), \( V^2(\mathbb{P}^2_Q) = [-2, \infty)[\mathbb{Z}\)

2. Use vanishing sets to find ideal \( I \) (along idea of G. Smith):
   \[
   I = B_{\Sigma}^{(44)} \equiv \langle x_0^{44}, x_1^{44}, x_2^{44} \rangle
   \]
Input and Output

- $C_{5-2} \subseteq \mathbb{P}_Q^2$
- $L_{5-2} \leftrightarrow M$ and $M$ defined by
  
  $S(-36) \oplus S(-39) \oplus S(-41) \oplus S(-23) \oplus S(-38) \rightarrow$
  
  $S(-6) \oplus S(-21) \rightarrow M \rightarrow 0$

$h^1\left(\mathbb{P}_Q^2, \tilde{M}\right) = ?$

Apply Algorithm

1. $V^0(\mathbb{P}_Q^2) = (-\infty, -1)_\mathbb{Z}$, $V^1(\mathbb{P}_Q^2) = \mathbb{Z}$, $V^2(\mathbb{P}_Q^2) = [-2, \infty)_\mathbb{Z}$
2. $I = B^{(44)}_\Sigma \equiv \langle x_0^{44}, x_1^{44}, x_2^{44} \rangle$
3. Compute presentation of $\text{Ext}^1_S{\left( B^{(44)}_\Sigma, M \right)}_0$:

$$\text{Ext}^1_S{\left( B^{(44)}_\Sigma, M \right)}_0$$
\( SU(5) \times U(1) \)-Tate model from 1706.04616

### Input and Output

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  \[ S(-6) \oplus S(-21) \rightarrow M \rightarrow 0 \]

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### Apply Algorithm

1. \( V^0(\mathbb{P}^2_Q) = (-\infty, -1]_\mathbb{Z}, \ V^1(\mathbb{P}^2_Q) = \mathbb{Z}, \ V^2(\mathbb{P}^2_Q) = [-2, \infty)_\mathbb{Z} \)
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   \[ Q^{37425} \rightarrow Q^{27201} \rightarrow \text{Ext}^1_S \left( B_{\Sigma}^{(44)}, M \right)_0 \rightarrow 0 \]
Input and Output

\[ C_{5-2} \subseteq \mathbb{P}^2_Q \]

\[ L_{5-2} \leftrightarrow M \text{ and } M \text{ defined by} \]
\[ S(-36) \oplus S(-39) \oplus S(-41) \oplus \]
\[ S(-23) \oplus S(-38) \rightarrow \]
\[ S(-6) \oplus S(-21) \rightarrow M \rightarrow 0 \]

\[ h^1 \left( \mathbb{P}^2_Q, \tilde{M} \right) = ? \]

Apply Algorithm

1. \[ V^0(\mathbb{P}^2_Q) = (-\infty, -1]_\mathbb{Z}, \quad V^1(\mathbb{P}^2_Q) = \mathbb{Z}, \quad V^2(\mathbb{P}^2_Q) = [-2, \infty)_\mathbb{Z} \]

2. \[ I = B^{(44)}_\Sigma \equiv \langle x_0^{44}, x_1^{44}, x_2^{44} \rangle \]

3. \[ \mathbb{Q}^{37425} \rightarrow \mathbb{Q}^{27201} \rightarrow \text{Ext}^1_S \left( B^{(44)}_\Sigma, M \right)_0 \rightarrow 0 \]

\[ \Rightarrow 28 = \dim_{\mathbb{Q}} \left[ \text{Ext}^1_S \left( B^{(44)}_\Sigma, M \right)_0 \right] = h^1 \left( \mathbb{P}^2_Q, \tilde{M} \right) \]
‘Scan’ Over Moduli Space

Note
Values of complex structure moduli enter definition of $M$.

Smoothness of matter curves NOT required

$\Rightarrow$ Run computation for different choices of moduli

$\text{SU}(5) \times U(1)$-Tate Model from 1706.04616 ($R = 5 - 2$)

$\tilde{a}_1, 0 \tilde{a}_2, 1 \tilde{a}_3, 2 \tilde{a}_4, 3 h i (C_R, L_R)$

$M_1 (x_1 - x_2)^4 x_7^1 x_10^2 x_13^3 (22, 43)$

$M_2 (x_1 - x_2) x_3^3 x_7^1 x_10^2 x_13^3 (21, 42)$

$M_3 x_4^3 x_7^1 x_7^2 (x_1 + x_2)^3 x_12^3 (11, 32)$

$M_4 (x_1 - x_2)^3 x_3^3 x_7^1 x_10^2 x_13^3 (9, 30)$

$M_5 x_4^3 x_7^1 x_7^2 (x_1 - x_2)^5 (6, 27)$

$M_6 x_4^3 x_7^1 x_10^2 x_8^3 (x_1 - x_2)^5 (7, 28)$

$M_7 x_4^3 x_7^1 x_9^2 (x_1 + x_2) x_10^3 (5, 26)$

$\text{Martin Bies}$

Zero Mode Counting in F-Theory via CAP
Values of complex structure moduli enter definition of $M$.
‘Scan’ Over Moduli Space

Note

- Values of complex structure moduli enter definition of $M$
- Smoothness of matter curves **NOT** required

\[
\begin{align*}
M_1 & \sim (x_1 - x_2)^4 x_7^1 x_10^2 x_13^3 (22, 43) \\
M_2 & \sim (x_1 - x_2) x_3^3 x_7^1 x_10^2 x_13^3 (21, 42) \\
M_3 & \sim x_4^3 x_7^1 x_12^3 (x_1 + x_2)^3 x_11^3 (11, 32) \\
M_4 & \sim (x_1 - x_2)^3 x_3^3 x_7^1 x_10^2 x_13^3 (9, 30) \\
M_5 & \sim x_4^3 x_7^1 x_8^2 (x_1 + x_2)^2 x_11^3 (7, 28) \\
M_6 & \sim x_4^3 x_7^1 x_9^2 (x_1 - x_2)^5 x_10^2 (6, 27) \\
M_7 & \sim x_4^3 x_7^1 x_9^2 (x_1 + x_2) x_10^3 (5, 26)
\end{align*}
\]
‘Scan’ Over Moduli Space

Note

- Values of complex structure moduli enter definition of $M$
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\begin{align*}
M_1 &= (x_1 - x_2)^4 x_7^1 x_{10}^2 x_{13}^3 (22, 43) \\
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M_3 &= x_4^3 x_7^1 x_7^2 (x_1 + x_2)^3 x_{12}^3 (11, 32) \\
M_4 &= (x_1 - x_2)^3 x_3^3 x_7^1 x_{10}^2 x_{13}^3 (9, 30) \\
M_5 &= x_4^3 x_7^1 x_8^2 (x_1 + x_2)^2 x_{11}^3 (7, 28) \\
M_6 &= x_4^3 x_7^1 x_{10}^2 x_8^3 (x_1 - x_2)^5 (6, 27) \\
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\end{align*}
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**Note**
- Values of complex structure moduli enter definition of $M$
- Smoothness of matter curves **NOT** required
⇒ Run computation for different choices of moduli

### $SU(5) \times U(1)$-Tate Model from 1706.04616 ($R = 5_{-2}$)

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<tr>
<th>$M_i$</th>
<th>$\tilde{a}_{1,0}$</th>
<th>$\tilde{a}_{2,1}$</th>
<th>$\tilde{a}_{3,2}$</th>
<th>$\tilde{a}_{4,3}$</th>
<th>$h_i (C_R, L_R)$</th>
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<td>$M_1$</td>
<td>$(x_1 - x_2)^4$</td>
<td>$x_1^7$</td>
<td>$x_2^{10}$</td>
<td>$x_3^{13}$</td>
<td>(22, 43)</td>
</tr>
<tr>
<td>$M_2$</td>
<td>$(x_1 - x_2)x_3^3$</td>
<td>$x_1^7$</td>
<td>$x_2^{10}$</td>
<td>$x_3^{13}$</td>
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<tr>
<td>$M_3$</td>
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<td>$x_1^7$</td>
<td>$x_2^7 (x_1 + x_2)^3$</td>
<td>$x_3^{12} (x_1 - x_2)$</td>
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<tr>
<td>$M_4$</td>
<td>$(x_1 - x_2)^3 x_3$</td>
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<td>$x_3^4$</td>
<td>$x_1^7$</td>
<td>$x_2^8 (x_1 + x_2)^2$</td>
<td>$x_3^{11} (x_1 - x_2)^2$</td>
<td>(7, 28)</td>
</tr>
<tr>
<td>$M_6$</td>
<td>$x_3^4$</td>
<td>$x_1^7$</td>
<td>$x_2^{10}$</td>
<td>$x_3^{10} (x_1 - x_2)^3$</td>
<td>(6, 27)</td>
</tr>
<tr>
<td>$M_7$</td>
<td>$x_3^4$</td>
<td>$x_1^7$</td>
<td>$x_2^9 (x_1 + x_2)$</td>
<td>$x_3^{10} (x_1 - x_2)^3$</td>
<td>(5, 26)</td>
</tr>
</tbody>
</table>
Summary and Conclusion

Have combined cohomCalg by R. Blumenhagen et al. (1003.5217, 1006.0780, 1006.2392, 1010.3717) and idea of G. Smith et al. (math/9807170, math/0305214, DOI: 10.4171/OWR/2013/25) ⇒

Toolkit to compute sheaf cohomologies of all coherent sheaves on toric varieties (visit https://github.com/HereAround)

Features:
- Count zero modes in 4d F-theory compactifications
- Matter curves need not be smooth, nor complete intersections!
- Of particular interest: hypercharge flux in F-theory GUTs (applications currently on their way)

Further possible applications
- Quite generally zero mode counting in topological string, IIB or heterotic compactifications
- T-branes as coherent sheaves (Collinucci et al. 1410.4178).

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Thank you for your attention!
From Divisors to Modules

Input and Output

\[ C = V(g_1, \ldots, g_k) \subseteq X_\Sigma \]
\[ D = V(f_1, \ldots, f_n) \in \text{Div}(C) \]

\[ \text{M s.t. supp(} \tilde{M} \text{) = } C \]
\[ \text{and } \tilde{M}|_C \cong \mathcal{O}_C(-D) \]
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**Step 1:**

\[ S(C) := S/\langle g_1, \ldots, g_k \rangle, \pi : S \rightarrow S(C) \]

\[ \bigoplus_{j \in J} S(C)(j) \xrightarrow{\ker(m)} \bigoplus_{i \in I} S(C)(i) \]

\[ m = (\pi(f_1), \ldots, \pi(f_n)) \]

\[ S(C) \xrightarrow{\sim} S(C) \]

\[ A_C \xrightarrow{l} S(C) \]
Step 2: Extend by zero to coherent sheaf on $X_\Sigma$

\[
\bigoplus_{j \in J} S(j) \xrightarrow{\ker(m)'} \bigoplus_{i \in I} S(i)
\]

\[
\bigoplus_{j \in J} S(j) \xrightarrow{\sim} A
\]

\[
\bigoplus_{k \in K} S(k)
\]

\[
\begin{pmatrix}
g_1 \\
\vdots \\
g_k
\end{pmatrix}
\]

\[
S(C)
\]

\[
\bigoplus_{k \in K} S(k) \xrightarrow{\sim} B
\]

\[
\Rightarrow M = A \otimes B \text{ satisfies } \text{Supp}(\tilde{M}) = C \text{ and } \tilde{M}|_C \cong \mathcal{O}_C(-D)
\]
Input and Output

\[ C = V(g_1, \ldots, g_k) \subseteq X_\Sigma \]
\[ D = V(f_1, \ldots, f_n) \in \text{Div}(C) \]

\[ M \text{ s.t. } \text{supp}(\tilde{M}) = C \]
\[ \text{and } \tilde{M}|_C \cong O_C(\mathcal{M}) \]

Strategy

1. Compute \( A_C \)
2. Dualise via \( A_C^\vee := \text{Hom}_{S(C)}(S(C), A_C) \)
3. Extend by zero by considering \( A^\vee \otimes B \)

\[ M^\vee := A^\vee \otimes B \text{ satisfies } \text{Supp}(\tilde{M}) = C \text{ and } \tilde{M}|_C \cong O_C(\mathcal{M}) \]
An idea of the sheafification functor

Affine open cover $\mathcal{U} = \{ \text{Spec } \mathbb{C}[x] \}_{\sigma \in \Sigma}$ with Cox ring $\mathbb{C}[x] \Rightarrow$ Covered by affine opens $\{ \text{Spec } \mathbb{C}[x] \}_{\sigma \in \Sigma}$

Localising ($\leftrightarrow$ restricting) a module $M \in \mathbb{C}$-fpgrmod $\Rightarrow M(\mathbb{C}[x])$ is f.p. $\mathbb{C}$-module

Consequence $M(\mathbb{C}[x])$ $\leftrightarrow$ coherent sheaf on $\text{Spec } \mathbb{C}[x]$

local sections: $\tilde{M}(\mathbb{C}[x])(D(f)) = M(\mathbb{C}[x]) \otimes \mathbb{C}[x] f$
An idea of the sheafification functor

Affine open cover

- Toric variety $X_\Sigma$ with Cox ring $S$
- Covered by affine opens $\left\{ U_\sigma = \text{Spec}_m(S(x^{\hat{\sigma}})) \right\}_{\sigma \in \Sigma}$
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**Affine open cover**
- Toric variety $X_\Sigma$ with Cox ring $S$
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**Localising (↔ restricting) a module**
- $M \in S\text{-fpgrmod}$
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### Affine open cover

- Toric variety $X_\Sigma$ with Cox ring $S$
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### Localising (↔ restricting) a module

- $M \in S\text{-fpgrmod}$
- $M_{(x^\sigma)}$ is f.p. $S_{(x^\sigma)}$-module

### Consequence

- $M_{(x^\sigma)} \leftrightarrow$ coherent sheaf on $U_\sigma = \text{Specm}(S_{(x^\sigma)})$
- Local sections: $\widetilde{M_{(x^\sigma)}}(D(f)) = M_{(x^\sigma)} \otimes_{S_{(x^\sigma)}} \left( S_{(x^\sigma)} \right)_f$
$$\dim_{\mathbb{Q}} \left[ \text{Ext}_S^0 \left( B^{(e)}_{\Sigma}, M_5 \right) \right] = 0$$
Module $M_5$ from 1706.04616: Quality Check I

$$\dim_{\mathbb{Q}} \left[ \operatorname{Ext}^0 \left( \mathcal{B}_{\Sigma}^{(e)}, M_5 \right) \right] = 0$$
Module $M_5$ from 1706.04616: Quality Check II

$$\dim_{\mathbb{Q}} \left[ \operatorname{Ext}^1_{\mathcal{S}} \left( B^{(e)}_{\Sigma}, M_5 \right) \right]_0$$

Graph showing the dependence of $\dim_{\mathbb{Q}} \left[ \operatorname{Ext}^1_{\mathcal{S}} \left( B^{(e)}_{\Sigma}, M_5 \right) \right]_0$ on $e$.
Module $M_5$ from 1706.04616: Quality Check II

$$\dim_{\mathbb{Q}} \left[ \text{Ext}^1_{\mathbb{Q}} \left( B_{\Sigma}^{(e)} , M_5 \right) \right] = 0$$

![Graph showing the dimension of Ext^1 over Q for B_{\Sigma}^{(e)} and M_5, with values ranging from 0 to 600 along the y-axis and from 0 to 48 along the x-axis.](image)
How to determine the ideal $I$ in step 2 of algorithm?

**Input**

- $M \in S$-fpgrmod
- $V^k(X_\Sigma) = \{ L \in \text{Pic}(X_\Sigma), h^k(X_\Sigma, L) = 0 \}$
How to determine the ideal $I$ in step 2 of algorithm?

**Input**
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**Preparation**
- $p \in \text{Cl}(X_\Sigma)$ **ample**, $m(p) = \{ m_1, \ldots, m_k \}$ all monomials of degree $p$ and $I(p, e) = \langle m_1^e, \ldots, m_k^e \rangle$
- Pick $e = 0$ and increase it until subsequent conditions are met
How to determine the ideal $I$ in step 2 of algorithm?

Input

- $M \in S$-fpgrmod
- $V^k(X_{\Sigma}) = \{L \in \text{Pic}(X_{\Sigma}), h^k(X_{\Sigma}, L) = 0\}$

How to find ideal $I$?

- Look at spectral sequence $E^{p,q}_2 \Rightarrow \text{Ext}_{O_{X_{\Sigma}}}^{p+q}(\widehat{\tilde{I}(p, e)}, \tilde{M})$
How to determine the ideal $I$ in step 2 of algorithm?

### Input

- $M \in S$-fpgrmod
- $V^k (X_\Sigma) = \{ L \in \text{Pic} (X_\Sigma), h^k (X_\Sigma, L) = 0 \}$

### How to find ideal $I$?

- Look at spectral sequence $E_2^{p,q} \Rightarrow \text{Ext}^{p+q}_{O_{X_\Sigma}} (\widetilde{I (p, e)}, \widetilde{M})$
- Some objects $E_2^{p,q}$ vanish as seen by $V^k (X_\Sigma)$
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**How to find ideal $I$?**

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- Some objects $\mathbb{E}^{p,q}_2$ vanish as seen by $V^k(X_\Sigma)$
- Does $\mathbb{E}^{p,q}_2$ degenerate (on $E_2$-sheet)? Is its limit (co)homology $H^m(C^0)$ of complex of global sections of vector bundles?
How to determine the ideal $I$ in step 2 of algorithm?

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$\Rightarrow$ If no – increase $e$ until this is the case!
How to determine the ideal $I$ in step 2 of algorithm?

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- Does $E_2^{p,q}$ degenerate (on $E_2$-sheet)? Is its limit (co)homology $H^m (\mathcal{C}^0)$ of complex of global sections of vector bundles?
  - => If no – increase $e$ until this is the case!
- Long exact sequence: sheaf cohomology $\leftrightarrow$ local cohomology
How to determine the ideal \( I \) in step 2 of algorithm?

**Input**

- \( M \in S\text{-fpgrmod} \)
- \( V^k (X_\Sigma) = \{ L \in \text{Pic} (X_\Sigma) \, , \, h^k (X_\Sigma, L) = 0 \} \)

How to find ideal \( I \)?

- Look at spectral sequence \( E^{p,q}_2 \Rightarrow \text{Ext}^{p+q}_{\mathcal{O}_{X_\Sigma}} (\sim (p, e), \tilde{M}) \)
- Some objects \( E^{p,q}_2 \) vanish as seen by \( V^k (X_\Sigma) \)
- Does \( E^{p,q}_2 \) degenerate (on \( E_2 \)-sheet)? Is its limit (co)homology \( H^m (\mathcal{C}^0) \) of complex of global sections of vector bundles?
  \( \Rightarrow \) If no – increase \( e \) until this is the case!
- Long exact sequence: sheaf cohomology \( \leftrightarrow \) local cohomology
  \( \Rightarrow \) Increase \( e \) further until \( H^m (\mathcal{C}^0) \cong \text{Ext}^m_S (I (p, e), \tilde{M})_0 \)