On micro-states of 4-d Black Holes and their stringy origin

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in memory of Yassen STANEV

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A great physicist, a wonderful colleague, a tender husband and father ... we will miss him a lot
Plan of the Talk

- Motivations: GW, ‘Black Hole’ mergers, BH dark matter ...
- BH Information Paradox and the Fuzz-ball Proposal
- 4-d BH micro-state geometries from string amplitudes
- L, K and M solutions from open string condensates at intersecting D3-branes
- Multi-center ansatz, Bubble equations and ‘regularity’
- Summary, conclusions and future directions
First, second and third ... direct detections of Gravitational Waves
Inspiral, merger, ring down ...
Intermediate-mass ‘Black Holes’ (∼ 50\(M_\odot\))
Stellar BH’s or Primordial BH’s?
‘Black Holes’ as Dark Matter?

Can Dark Matter be made of ‘intermediate-mass’ (10-1000 $M_\odot$) primordial BH’s?... Probably Not [E. Mediavilla et al 2017]

Effect of distribution of masses on light from distant quasars: micro-lensing objects 0.5-4.5 $10^{-1}M_\odot$, only 20% of total matter such as ‘normal’ stellar matter
Information Paradox

- Pure state enters into a BH
- Emitted radiation is thermal (no information), but entangled with BH.
- Emitted particles do not depend on the state of earlier produced particles ...
- BH completely evaporates: there is nothing to be entangled with.
- ... only radiation in a mixed state $\Rightarrow$ unitarity is lost!
The paradox cannot be solved by adding small corrections to the semi-classical computation and information cannot be recovered at the last stages of evaporation.

- Loss of unitarity [Hawking]

- Remnants, Baby Universe [Susskind]

- Non Local BH-radiation interactions [Maldacena-Susskind, Raju-Papadodimas]

- Hairs in the asymptotic structure of space-time [Hawking, Perry, Strominger; Dvali, Gomez, Lüst], ...

- The ‘horizon’ is no more in an “information free vacuum” [Lunin, Mathur]

We will explore the last possibility. Rather than only solving an ad hoc problem, this resolution emerges naturally from String Theory, fitting into a bigger picture for Quantum Gravity.

Every (BPS) Black-Hole micro-state is dual to a smooth, horizon-less (super)gravity solution. NO singularity (caveat) Quantum Gravity effects are horizon-sized due to huge phase space. Would-be horizon carries information ... the paradox is solved.

Far away fuzz-ball resembles a BH: every micro-state has the same asymptotic charges \( (M, J, Q) \) as the would-be BH. The boundary of the region where micro-states differ from BH satisfy \( S \approx A/4 \). [S. Mathur (2005)]

Classical BH arises as “coarse-grained” description when only the geometry outside the “horizon” is taken into account.
BH’s in String Theory: the D1-D5-P paradigm

- Strong Coupling $g_s Q \gg 1$: ‘large’ BPS Black Hole in $D = 5$, small curvature at the horizon

$$ ds^2 = (H_1 H_5)^{-1/2}[-dt^2 + dy_5^2 + (H_P - 1)(dt + dy_5)^2] $$

$$ + (H_1 H_5)^{1/2}(dx_1^2 + \ldots dx_4^2) + H_1^{1/2} H_5^{-1/2}(dy_6^2 + \ldots dy_9^2) $$

Macroscopic (geometric) entropy $S_{BH} = 2\pi \sqrt{Q_1 Q_5 Q_P}$

- Weak Coupling $g_s Q \ll 1$: D-branes and open strings
  For $V_{T_4} \ll R_{S_1}^4$, $\mathcal{N} = (4, 4)$ $U(Q_1) \times U(Q_5)$ theory in $D = 2$
  with $c = n_{bose} + \frac{1}{2} n_{fermion} = 6Q_1 Q_5$, from $(1, 5)$ strings.
  For large charges, degeneracy given by C(H)ardy-Ramanujan
  formula: $d(Q_P) \sim e^{2\pi \sqrt{c Q_P}/6} \Rightarrow S_{micro} = \log d(Q_P)$

For BPS BH’s in $D = 5$: $S_{micro} = S_{MACRO}$ [Strominger, Vafa (1996)]

But what are the micro-states in the gravity picture?
D1-D5 Fuzz-ball

\[ ds^2 = (H_1 H_5)^{-1/2}[-(dt + A_i dx^i)^2 + (dy_5 + B_i dx^i)^2] \]

\[ + (H_1 H_5)^{1/2} \sum_{i=1}^{4} dx_i^2 + (H_1 / H_5)^{1/2} \sum_{a=1}^{4} dy_a^2 \]

\[ H_1 = 1 + \frac{Q_1}{\ell} \int_0^{\ell} \frac{dv}{|\vec{x} - \vec{F}(v)|^2} \quad H_5 = 1 + \frac{Q_1}{\ell} \int_0^{\ell} \frac{dv |\dot{F}(v)|^2}{|\vec{x} - \vec{F}(v)|^2} \]

\[ A_i = \frac{Q_1}{\ell} \int_0^{\ell} \frac{dv \dot{F}_i(v)}{|\vec{x} - \vec{F}(v)|^2} \quad dB = \star_4 dA \quad v = t - y_5 \]

E.g. circle: \( F_1 = \cos(2\pi v / \ell), \quad F_2 = \sin(2\pi v / \ell), \quad F_3 = F_4 = 0 \)

Coordinate singularity along \( x^i = F^i(v) \), resolved: K-K monopole

Throat ends in a smooth “cap”, shape determined by \( F(v) \) profile

Entropy \( S_{\text{micro}} = 2\sqrt{2\pi} \sqrt{Q_1 Q_5} \) from CFT or from ‘geometric quantization’ of transverse ‘string’ oscillations (in F1-P frame)

Fuzz-ball proposal ‘proven’ in the 2 charge case, yet ‘small’ BH’s ‘Large’ BH’s require 3 charges in \( D = 5 \) or 4 charges in \( D = 4 \).
Part II
4-d BH micro-state geometries
from string amplitudes
Stringy Origin of 4d BPS Black Holes Micro-states

Enormous progress in 5-d [Bena, Giusto, Gibbons, Martinec, Russo, Shigemori, Warner, ...]  
Much less known in 4-d!  
Our goal: recover micro-state geometries from the underlying fundamental string theory description  
We consider bound-states of 4 stacks of (orthogonally) intersecting D3-branes on $T^6$ in Type IIB ... dual to D2-D2-D2-D6 in Type IIA or M2-M5-P-KK6 in M-theory

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We derive a 1:1 relation between open string condensates and (super)gravity fields in the bulk for a large class of 4d BPS BH’s
Mixed Open-Closed String Amplitudes

micro-state geometries derived from mixed open-closed disk amplitudes, computing the emission rate of massless closed strings from open string condensates binding different stacks of branes.

Closed String Fields

\[ g_{MN}, b_{MN}, C^{(4)}_{MNPQ} \]

Open String Fields

\[ \mu^A, \phi^i \]
From String Amplitudes to Supergravity Fields

We work at leading order in $g_s$ (disk), take all open string momenta equal (or tending) to zero and closed string momentum $k$ only in non compact space directions.

$$\mathcal{A}(h, k) \propto \int \frac{d^{2+n}z}{V_{CKV}} \langle W_{\text{closed}}(h, k; z, \bar{z}) V_{\text{open}}(z_1) \ldots V_{\text{open}}(z_n) \rangle$$

Relation is well defined only if disk diagram cannot factorize via the exchange of open-string states
Choose ‘polarizations’ of open strings in such a way that NO factorization diagram be allowed
The deviation from flat space of a closed-string field

$$\delta \tilde{\phi}(k) = -\frac{i}{k^2} \frac{\delta \mathcal{A}(h, k)}{\delta h} \quad \rightarrow \quad \delta \phi(x) = \int \frac{d^3k}{(2\pi)^2} \tilde{\phi}(k) e^{ikx}$$
Supergravity Solution: the Love-ful Eight

Type IIB supergravity equations (with $\phi = g_s$, $C_0 = C_2 = B_2 = 0$)

$$R_{MN} = \frac{1}{4 \cdot 4!} F_{MP_1 P_2 P_3 P_4} F_{N P_1 P_2 P_3 P_4}$$

$$F_5 = *_{10} F_5$$

$$F_5 = dC_4$$

8 harmonic functions $H_a = \{V, L_I, K^I, M\}, I = 1, 2, 3$ (STU model)

$$ds^2 = -e^{2U} (dt + w)^2 + e^{-2U} |d\vec{x}|^2 + \sum_{I=1}^{3} \left[ \frac{dy_I^2}{Ve^{2U} Z_I} + Ve^{2U} Z_I \tilde{e}_I^2 \right]$$

$$C_4 = \alpha_0 \cdot \tilde{e}_1 \cdot \tilde{e}_2 \cdot \tilde{e}_3 + \beta_0 \cdot dy_1 \cdot dy_2 \cdot dy_3 + \frac{\epsilon_{IJK}}{2} \left( \alpha_I \cdot dy_I \cdot \tilde{e}_J \cdot \tilde{e}_K + \beta_I \cdot \tilde{e}_I \cdot dy_J \cdot dy_K \right)$$

where $\cdot = \wedge$, $\epsilon_{IJK}$ (reduced) intersection form for 3-cycles in $T^6$,

$$Z_I = L_I + \left| \epsilon_{IJK} \right| \frac{K^J K^K}{V}$$

$$\mu = \frac{M}{2} + \frac{L_I K^I}{2 V} + \left| \epsilon_{IJK} \right| \frac{K^I K^J K^K}{6 V^2}$$

$$e^{-4U} = Z_1 Z_2 Z_3 V - \mu^2 V^2$$

$$*_{3} dw = V d\mu - \mu dV - VZ_1 d\frac{K^I}{V}$$

$$\tilde{e}_I = d\tilde{y}_I - \left( \frac{K^I}{V} - \frac{\mu}{Z_I} \right) dy_I$$
L solutions

L solutions are geometries that fall-off at infinity as $Q_i/r$, corresponding to a single stack of branes e.g.

$$V = L(x) \quad M = K^I = 0 \quad L_I = 1$$

At linear order in $\ell_{D3} \sim g_s \sqrt{\alpha'}$ one finds:

$$\delta g_{MN} dx^M dx^N = \frac{\delta L}{2} \left[ dt^2 - \sum_{i=1}^{3} (dy_i^2 - dx_i^2 - d\tilde{y}_i^2) \right] + \ldots$$

$$\delta C_4 = -\delta L \wedge dt \wedge dy_1 \wedge dy_2 \wedge dy_3 + A \wedge d\tilde{y}_1 \wedge d\tilde{y}_2 \wedge d\tilde{y}_3 + \ldots$$

with $\delta L = L - 1$ and $A$ both of order $\ell_{D3}$. One can take:

$$L = 1 + \frac{\ell_{D3} N_0}{|x|} + \ldots \quad *_3 dL = dA$$
One-boundary Amplitude

Very well known result, modulo ‘untwisted’ open-string insertions

\[ \mathcal{A}_{NS-NS,\xi(\phi)} = \left\langle c\bar{c}W_{NS-NS}^{(-1,-1)}(z, \bar{z})cV_{\xi(\phi)}^{(0)}(z_1) \right\rangle = i \ c_{NS} \ \text{tr}(ER)\xi(k) \]

where \( E = h + b \), \( R \) reflection matrix (+1 Neumann, −1 Dirichlet)

\[ W_{NSNS}^{(-1,-1)}(z, \bar{z}) = c_{NS} \ (ER)_{MN} e^{-\varphi \psi^M} e^{ikX(z)} e^{-\varphi \psi^N} e^{ikRX(\bar{z})} \]

\[ V_{\xi(\phi)}^{(0)}(z_1) = \sum_{n=0}^{\infty} \xi_{i_1...i_n} \partial X^{i_1}(z_1) \prod_{a=2}^{n} \int_{-\infty}^{\infty} \frac{dz_a}{2\pi} \partial X^{i_a}(z_a) \]

with \( \xi(\phi) = \sum_{n=0}^{\infty} \xi_{i_1...i_n} \phi^{i_1} \ldots \phi^{i_n} \) and \( z_a = \bar{z}_a \) (open strings)

The asymptotic deviation from the flat metric

\[ \delta \tilde{g}_{MN}(k) = \left( -\frac{i}{k^2} \right) \sum_{n=0}^{\infty} \frac{\delta \mathcal{A}_{NS-NS,\phi^n}}{\delta h_{MN}} = c_{NS} \frac{\xi(k)}{k^2} \ (\eta R)_{MN} \]

After Fourier transform one finds agreement with SUGRA

\[ \delta g_{MN} = \int \frac{d^3k}{(2\pi)^3} \delta \tilde{g}_{MN} = -\frac{1}{2} (\eta R)_{MN} \delta L(x) \quad \text{and} \quad \delta b_{MN} = 0 ! \]

In particular, for a single D3-brane at position \( x = a \):

\[ \xi(\phi) \sim e^{ia\phi} \]
K solutions

K solutions are geometries that fall-off at infinity as $Q_i Q_j / r^2$ e.g.

$$K^3 = -M = K(x) \quad \mu = 0 \quad L_1 = V = 1 \quad K^1 = K^2 = 0$$

Associated to fermionic bilinears localized at the intersection of two branes and in general carry angular momentum.

At linear order in $\ell_{D3}$ one finds $(\ast_3 dw = -dK)$:

$$\delta g_{MN} dx^M dx^N = -2 w dt - 2 K dy_3 d\tilde{y}_3 + \ldots$$

$$\delta C_4 = (K dt \wedge dy_3 - w \wedge d\tilde{y}_3) \wedge (dy_1 \wedge d\tilde{y}_2 + d\tilde{y}_1 \wedge dy_2)$$

For example one can take

$$K \approx \frac{v_i x_i}{|x|^3} \quad w \approx \epsilon_{ijk} v_i \frac{x_j dx_k}{|x|^3}$$
Two-boundary Amplitude

\[ \mathcal{A}_{\mu^2, \xi(\phi)}^{NS-NS} = \int d\tau^4 \langle c(z_1) V_{\bar{\mu}}(z_1) c(z_2) V_{\mu}(z_2) c(z_3) W(z_3, z_4) V_{\xi}(\phi) \rangle \]

where

\[ V_{\bar{\mu}}(z_1) = \bar{\mu}^A e^{-\varphi/2} S_A \sigma_2 \sigma_3 \]

\[ V_{\mu}(z_2) = \mu^B e^{-\varphi/2} S_B \sigma_2 \sigma_3 \]

\[ \left\langle \text{tr} \bar{\mu}^{(A} \mu^{B)} \right\rangle = \frac{1}{3!} \nu^{MNP} \Gamma^{AB}_{MNP} \]

\[ \mathcal{A}_{\mu^2, \xi(\phi)}^{NS-NS} = \frac{\xi(k)}{3!} (ER)_{MN} k_P \nu^{MNP} \]

with \( \nu^{MNP} \in 10 \) of \( \text{SO}(6) \) (NO 6!!) e.g. for \( \nu_{3y_3\tilde{y}_3} = -\nu_{12t} = 4\pi \nu \)

\[ \delta g_{2t} = -\nu \frac{x_1}{|x|^3} \]

\[ \delta g_{1t} = \nu \frac{x_2}{|x|^3} \]

\[ \delta g_{y_3\tilde{y}_3} = -\nu \frac{x_3}{|x|^3} \]
M solutions

M solutions are geometries that fall-off at infinity as \( Q_1 Q_2 Q_3 Q_4 / r^3 \)
e.g.

\[
K^2 = M = M(x) \quad \mu = M \quad L_1 = V = 1 \quad K^1 = K^3 = 0
\]

\[
\delta g_{MN} dx^M dx^N = 2M \left( dy_1 d\tilde{y}_1 + dy_3 d\tilde{y}_3 \right) + \ldots
\]

\[
\delta C_4 = -M dt \wedge \left( dy_1 \wedge d\tilde{y}_2 \wedge dy_3 + d\tilde{y}_1 \wedge d\tilde{y}_2 \wedge d\tilde{y}_3 \right)
+ w_2 \wedge \left( dy_1 \wedge dy_2 \wedge dy_3 + d\tilde{y}_1 \wedge dy_2 \wedge d\tilde{y}_3 \right) + \ldots
\]

with \( w_2 = \ast_3 dM \)

In particular one can take the harmonic \( M \) to be a ‘quadru-pole’

\[
M \approx \nu_{ij} \frac{3 x_i x_j - \delta_{ij}|x|^2}{|x|^5}
\]
Four-Boundary Amplitude

Insertion of four fermions $\mu_{a,a+1}$ starting on D3-branes of type $a$ and ending on D3-branes of type $a+1$ with $a = 0, 1, 2, 3 \text{ (mod 4)}$ Even if each intersection preserves $\mathcal{N} = 2$ SUSY (1/4 BPS), so that each fermion $\mu_{a,a+1}$ paired with its conjugate $\bar{\mu}_{a,a+1}$, whole configuration preserves only $\mathcal{N} = 1$ SUSY (1/8 BPS). The condensate is complex e.g. $\mu_1 \mu_2 \bar{\mu}_3 \bar{\mu}_4 \neq \bar{\mu}_1 \bar{\mu}_2 \mu_3 \mu_4$

$$\mathcal{A}_{\mu^4,\xi(\phi)}^{NS-NS} = \int dz d^2 w \langle cV_{\mu_1}(z_1) cV_{\mu_2}(z_2) V_{\bar{\mu}_3}(z=\bar{\bar{z}}) cV_{\bar{\mu}_4}(z_4) W_{NSNS}(w, \bar{w}) V_{\xi(\phi)} \rangle$$
Four-Boundary Amplitude

\[
\left\langle \text{tr} \mu_1^{(\alpha \mu_2)} \bar{\mu}_3^{(\bar{\alpha} \bar{\mu}_4)} \right\rangle = \frac{2\pi \nu^{ij}}{c_{\text{NS}} I_1} \sigma_i^{\alpha \bar{\alpha}} \bar{\sigma}_j^{\beta \bar{\beta}} \quad \nu^{ij} \in (3, 3) \text{ of } SU_L(2) \times SU_R(2)
\]

Need \( Z_2 \) twist field correlator on the boundary of the disk

\[
\left\langle \sigma_2(z_1)\sigma_2(z_2)\sigma_2(z_3)\sigma_2(z_4) \right\rangle = f \left( \frac{Z_{14}Z_{23}}{Z_{13}Z_{24}} \right) \left( \frac{Z_{13}Z_{24}}{Z_{12}Z_{23}Z_{34}Z_{41}} \right)^{1/4}
\]

where \( f(x) = \frac{\Lambda(x)}{\sqrt{F(x)F(1-x)}} \) with \( F(x) = 2 F_1(1/2, 1/2; 1; x) \) and

\[
\Lambda(x) = \sum n_1, n_2 \exp \left\{ -\frac{2\pi}{\alpha'} \left[ \frac{F(1-x)}{F(x)} n_1^2 R_1^2 + \frac{F(x)}{F(1-x)} n_2^2 R_2^2 \right] \right\} \approx 1
\]

\[
\mathcal{A}_{\mu^4, \xi(\phi)}^{\text{NS-NS}} = \left[ (ER)_{[11]} + (ER)_{[33]} \right] k_i k_j \nu^{ij} \xi(k)
\]

so that \( \delta \tilde{g}_{1\bar{1}} = \delta \tilde{g}_{3\bar{3}} = -2\pi i \xi(k) \nu^{ij} k_i k_j / k^2 \)

Agreement with SUGRA solution to leading order in \( \ell_{D3} \).
... Some speculations on the entropy

- Thanks to $\mathcal{N} = 2$ SUSY preserving $D3_a D3_b$ intersections, $D3^4$ more closely related than $D1D5P$ to $D1D5$ system. ‘Realistic’ four-charge case may turn out to be simpler than three-charge case!

- The number of disks with four different boundaries grows as $Q_1 Q_2 Q_3 Q_4 = \mathcal{I}_4$. One can attempt the calculation of the entropy via geometric quantization by introducing suitable profile-dependent harmonic functions, as in the D1-D5 case.

- A family of asymptotically $AdS_2 \times S^2 \times T^6$ geometries has been found and shown to be regular. Harmonic functions written in terms of an arbitrary profile [Lunin (2015)]

$$H(\vec{x}) = h_{\text{reg}}(\vec{x}) + \int_0^{2\pi} \frac{dv}{2\pi} \frac{1}{|\vec{x} - \vec{F}(v)|} \sqrt{1 + \frac{(\vec{x} - \vec{F}) \cdot \vec{A}(v)}{|\vec{x} - \vec{F}|^2}}$$

- For asymptotically flat solutions in 4d, no-go theorem: NO non-singular solutions in GR. Either include higher-derivative terms or get ‘generalised’ regularity in five or higher dimension.
Part III.
Multi-center ansatz, Bubble Equations boundary conditions and ‘regularity’
From 4 to 10 (or 11) dimensions and back: STU et cetera

4-dim $\mathcal{M}_{STU} = [SL(2, R)/U(1)]^3 \subset E_{7(+7)}/SU(8) = \mathcal{M}_{N=8}$

$L_{STU \sim U_1 U_2 U_3} = \frac{1}{16\pi G} \left(R_4 - \sum_{l=1}^{3} \frac{\partial_\mu U_l \partial^{\mu} \bar{U}_l}{2 \text{Im} U_l^2} - \frac{1}{4} F_a \mathcal{I}^{ab} F_b - \frac{1}{4} F_a \mathcal{R}^{ab} \tilde{F}_b \right)$

10-dim uplift

$ds^2_{10} = -e^{2U}(dt + w)^2 + e^{-2U}|d\vec{x}|^2 + \sum_{l=1}^{3} \left[ \frac{dy_i^2}{Ve^{2U} Z_i} + Ve^{2U} Z_i \tilde{e}_i^2 \right]$

where $Z_i = L_i + \frac{|\epsilon_{IJK}|}{2} \frac{K^J K^K}{V}$, $\mu = \frac{M}{2} + \frac{L_i K^I}{2 V} + \frac{|\epsilon_{IJK}|}{6} \frac{K^I K^J K^K}{V^2}$ and $e^{-4U} = I_4(L_i, V, K^I, M) = Z_1 Z_2 Z_3 V - \mu^2 V^2 = L_1 L_2 L_3 V - K^1 K^2 K^3 M$

$+ \frac{1}{2} \sum_{I>J}^{3} K^I K^J L_I L_J - \frac{1}{2} MV \sum_{I=1}^{3} K^I L_I - \frac{1}{4} M^2 V^2 - \frac{1}{4} \sum_{I=1}^{3} (K^I)^2 L_I^2$

11-dim uplift $ds^2_{T^6} = \sum_{I=1}^{3} Z^{-1}_I (Z_1 Z_2 Z_3)^{\frac{1}{3}} (dy_i^2 + d\tilde{y}_i^2)$ and

$ds^2_{5} = -\frac{[dt + \mu(d\Psi + w_0) + w]^2}{(Z_1 Z_2 Z_3)^{\frac{2}{3}}} + (Z_1 Z_2 Z_3)^{\frac{1}{3}} \left[ V^{-1}(d\Psi + w_0)^2 + V|d\vec{x}|^2 \right]$
Asymptotic geometry and charges

Using (asymptotic) Killing vectors (later on $16\pi G = 1$)

$$\mathcal{M} = \frac{1}{8\pi G} \int_{S^2_\infty} \star_4 d\xi(t), \quad J = -\frac{1}{16\pi G} \int_{S^2_\infty} \star_4 d\xi(\phi),$$

$$Q^a = \frac{1}{4\pi} \int_{S^2_\infty} (\mathcal{I}^{ab} \star_4 F_b - \mathcal{R}^{ab} F_b), \quad P_a = \frac{1}{4\pi} \int_{S^2_\infty} F_a$$

Boundary conditions and charges for orthogonal branes

$$V \approx 1 + \frac{v}{r} \quad L_I \approx 1 + \frac{\ell_I}{r} \quad K^I = M \approx 0$$

$$\mathcal{M} = v + \ell_1 + \ell_2 + \ell_3, \quad P = (v, 0, 0, 0), \quad Q = (0, \ell_1, \ell_2, \ell_3), \quad J = 0$$

Boundary conditions and charges for branes at angle

$$V \approx 1 + \frac{v}{r} \quad L_I \approx 1 + \frac{\ell_I}{r} \quad K^1 \approx g + \frac{k^1}{r} \quad K^2 \approx g \quad K^3 = M = 0$$

$$\mathcal{M} = v + \ell_1 + \ell_2 + \ell_3, \quad P = (v, -g(\ell_1 + \ell_2), 0, 0), \quad Q = (0, \ell_1, \ell_2, \ell_3), \quad J = 0$$
Micro-state geometries

Multi-center Taub-NUT ansatz \( (r_i = |\vec{x} - \vec{x}_i|, \ i = 1, \ldots N) \)

[\text{Bena, Warner, Gibbons, Cvetic, Lu, Pope, ...}]

\[
V = v_0 + \sum_{i=1}^{N} \frac{q_i}{r_i} \quad L_I = \ell_{0I} + \sum_{i=1}^{N} \frac{\ell_{I,i}}{r_i}
\]

\[
K^I = k_0^I + \sum_{i=1}^{N} \frac{k_i^I}{r_i} \quad M = m_0 + \sum_{i=1}^{N} \frac{m_i}{r_i}
\]

Near each center, \( R^4 / Z_{|q_i|} \), asymptotically \( R^3 \times S_\psi^1 \)
Geometry factorises, i.e. regular in 5-d (!), if near the centers

\[
Z_I \big|_{r_i \approx 0} \approx \zeta_i^i \ (\text{finite}) \quad \text{and} \quad \mu \big|_{r_i \approx 0} \approx 0 \ (\text{zero})
\]

Absence of horizons and closed time-like curves requires

\[
Z_I V > 0 \quad \text{and} \quad e^{2U} > 0
\]

\( w \) closed exact form near the centres
Bubble equations

$Z_i$ finite near the centers if

$$\ell_{I,i} = -\frac{1}{2} \frac{|\epsilon_{IJK}|}{q_i} k_i^J k_i^K, \quad m_i = \frac{k_i^1 k_i^2 k_i^3}{q_i^2}$$

$\mu$ vanishes near the centers if Bubble Equations are satisfied

$$\sum_{j=1}^{N} \frac{\Pi_{ij}}{r_{ij}} + v_0 \frac{k_i^1 k_i^2 k_i^3}{q_i^2} - \sum_{l=1}^{3} \ell_{0l} k_i^l - |\epsilon_{IJK}| \frac{k_0^l k_i^J k_i^K}{2 q_i} - m_0 q_i = 0$$

with $\Pi_{ij} = (q_i q_j)^{-2} \prod_{l=1}^{3} \left( k_i^l q_j - k_j^l q_i \right)$ and $r_{ij} = |\vec{x}_i - \vec{x}_j|$.

Bubble equations imply absence of pernicious Dirac-Misner strings

$$*_3 dw = \frac{1}{2} \sum_{i,j=1}^{N} \Pi_{ij} \left( \frac{1}{r_j} - \frac{1}{r_{ij}} \right) d \frac{1}{r_i} = \frac{1}{4} \sum_{i,j=1}^{N} \Pi_{ij} \omega_{ij}$$

with $\omega_{ij} = (\vec{n}_i + \vec{n}_{ij}) \cdot (\vec{n}_j - \vec{n}_{ij}) d\phi_{ij}/r_{ij}$ free of D-M strings along lines connecting two centers, since numerator vanishes there.
Scaling solutions

If the coefficients $k_i^l$ satisfy

$$v_0 \ m_i - \sum_{l=1}^{3} \ell_{0l} \ k_i^l + k_0^l \ l_i - m_0 q_i = 0$$

invariance under rigid rescaling of the positions of the centres

$$\vec{x}_i \to \lambda \vec{x}_i$$

Multiplying (...) by the positions of the centers $\vec{x}_i$, the solution can be shown to carry zero angular momentum

$$\vec{J} = m_0 \ \vec{v}_2 - v_0 \ \vec{m}_2 + \ell_{0l} \ \vec{k}_2^l - k_0^l \ \vec{l}_{2l} = 0$$

in agreement with (Sen’s) expectations for individual micro-states
Fuzz-balls of orthogonal branes

Boundary conditions

\[ \ell_{0I} = v_0 = 1 \quad m_0 = m = k_0^I = k^I = 0 \]

For \( q_i = 1 \) (to avoid orbifold singularities, for simplicity)

\[ P_0 = N \quad , \quad Q_I = - \sum_{i=1}^{N} \frac{|\epsilon_{IJK}| k^J_i k^K_i}{2} \]

Bubble Equations (\( q_i = 1! \))

\[ \sum_{j \neq i}^{N} \prod_{l=1}^{3} \frac{(k^I_i - k^I_j)}{r_{ij}} + k^1_i k^2_i k^3_i - \sum_{l=1}^{3} k^l_i = 0 \]

absence of horizons and of closed time-like curves requires

\[ Z_I V > 0 \quad and \quad e^{2U} > 0 \]

Configurations with one or two centers fail to meet the BPS requirement \( Q_I > 0 \). Let us start (and end) with three centers
3-center case \( N = 3 = P_0 \)

\[
k'_i = \begin{pmatrix} -n_1 n_2 & -n_1 n_3 & n_1 (n_2 + n_3) \\ n_3 & n_2 & -n_2 - n_3 \\ -n_4 & n_4 & 0 \end{pmatrix}
\]

scaling solutions: \( n_2 = 0, n_1 = 1, n_3 = n_4 = n \)

\( Q_1 = Q_2 = Q_3 = n^2 \), any \( r_{12} = r_{23} = r_{13} = R \)

non-scaling solutions:

\( n_2 = 0, n_1 = n_3 = 1, n_4 = n: r_{13} = r_{23} = R \) undetermined

\( Q_1 = Q_2 = n^n Q_3 = 1 
\)

\( r_{12} = \frac{2 n r_{23}}{2 n + (n - 1) r_{23}} \)

\( n_2 = n_4 = n, n_1 = 1, n_3 = 2 n: r_{13} = r_{23} = R \) undetermined

\( Q_1 = Q_2 = n^2 Q_3 = 13 n^2 
\)

\( r_{12} = \frac{r_{23}}{10 + r_{23}} \)

\( n_2 = 0, n_1 = 3 n, n_3 = 2 n, n_4 = n: r_{23} < 6 (2 - \sqrt{2}) n^2 \)

\( Q_1 = 2 n^2, Q_2 = 6 n^2, Q_3 = 3 n^2, r_{12} = \frac{12 n^2 r_{23}}{12 n^2 - r_{23}}, r_{13} = \frac{6 n^2 r_{23}}{6 n^2 - r_{23}} \)
Fuzz-balls of branes at angle

New boundary conditions

$$\ell_0 I = v_0 = 1, m_0 = m = k_0^3 = k^3 = k^2 = 0, k_0^1 = k_0^2 = g, k^1 = g(\ell_1 + \ell_2)$$

Generalized bubble equations

$$\sum_{j \neq i}^{N} \frac{k_{ij}^{(1)} k_{ij}^{(2)} k_{ij}^{(3)}}{r_{ij}} + k_i^1 k_i^2 k_i^3 - \sum_{l=1}^{3} k_i^l - g k_i^2 k_i^3 - g k_i^1 k_i^3 = 0$$

3-center case, $P_0 = 3$, $n_1, n_2, n_3$ positive integers, $g$ rational

$$k_i^l = \begin{pmatrix} 0 & -n_1 & n_1 + g n_3 (n_1 + n_2) \\ n_2 & 0 & -n_2 \\ -n_3 & n_3 & 0 \end{pmatrix}$$

$$Q_1 = n_2 n_3, Q_2 = n_1 n_3, Q_3 = n_1 n_2 + g n_2 n_3 (n_1 + n_2)$$
Future directions
Future Directions

- Generalize to D3-brane configurations with generic tilting on orbifolds (e.g. $T^6/Z_3$)
- Compute the contribution to the entropy of the known configurations (scaling vs non-scaling) and understand their CFT (AdS) and/or Quiver Quantum Mechanics description
  [Denef, Pioline, Manschoot, Sen, Garavuso, ... Morales, Pieri, Russo w.i.p.]
- Apply similar techniques to scattering of closed string (massive) states [Garousi, Myers, Klebanov, Hashimoto, D’Appollonio, Di Vecchia, Russo, Veneziano, Turton, MB, Teresi, ...] ... work in progress, stay tuned
- Construct new micro-state SUGRA solutions corresponding to different choices of the open-string condensates
- Find ‘regular’ non extremal and realistic (four-charge) geometries
- Study fuzz-ball mergers and GW production ... experimental test of String Theory ?